3) Sequences and series

3.1) Arithmetic sequences

Chapter CONTENTS

Worked example

The nth term of an arithmetic sequence is $u_n = 35 - 3n$.

a) Write down the first 3 terms of the sequence.

b) Find the first term in the sequence that is negative.

Your turn

The nth term of an arithmetic sequence is $u_n = 55 - 2n$.

a) Write down the first 3 terms of the sequence.

b) Find the first term in the sequence that is negative.

a) $u_1 = 53, u_2 = 51, u_3 = 49$

b) $u_{28} = -1$

Worked example

Find the nth term of each arithmetic sequence.

a) $-6, 2, 10, 18, 26, \ldots$

b) $788, 785, 782, 779, 886, \ldots$

Your turn

Find the nth term of each arithmetic sequence.

a) $6, 20, 34, 48, 62, \ldots$

b) $101, 94, 87, 80, 73, \ldots$

a) $u_n = 14n - 8$

b) $u_n = 108n - 7$

Worked example

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.

Given that $u_5 = 17$ and $u_9 = 33$, find the values of the constants a and b.

Your turn

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.

Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

$a = 3, b = -4$

Worked example

For which values of x would the expression $-2, 4x^2$ and $17x$ form the first three terms of an arithmetic sequence?

Your turn

For which values of x would the expression $-8, x^2$ and $17x$ form the first three terms of an arithmetic sequence?

$$x = \frac{1}{2}, x = 8$$

Worked example

An arithmetic sequence has first term k^2 and common difference k, where $k < 0$. The third term of the sequence is 24. Find the value of k

Your turn

An arithmetic sequence has first term k^2 and common difference k, where $k > 0$. The fourth term of the sequence is 40. Find the value of k

$k = 5$

Worked example	Your turn
Is 100 in the sequence: $-3, 4, 11, 18, \ldots$?	Is 100 in the sequence: $4, 7, 10, 13, \ldots$? Yes – the 33rd term
Is 10 in the sequence: $127\ 118, 109, 100, \ldots$?	Is 10 in the sequence: $85, 78, 71, 64 \ldots$? No $-7n + 92 = 10$ solves to give $n = \frac{82}{7}$ which is not an integer.

Worked example	Your turn
The first five terms of each sequence are shown. Find two numbers which are in both sequences. $3, 10, 17, 24, 31, \ldots$ $-4, -1, 2, 5, 8, \ldots$	The first five terms of each sequence are shown. Find two numbers which are in both sequences. $2, 7, 12, 17, 22, \ldots$ $-4, -1, 2, 5, 8, \ldots$ $2, 27$

Worked example	Your turn
Find the n^{th} term of the sequence $\frac{1}{3}, \frac{4}{5}, \frac{7}{7}, \frac{10}{9}, \ldots$	Find the n^{th} term of the sequence $\frac{1}{3}, \frac{3}{6}, \frac{5}{9}, \frac{7}{12}, \ldots$ $\frac{2n-1}{3n}$

Worked example	Your turn
The fifth term of an arithmetic sequence is 9. The twelfth term of the same arithmetic sequence is 23. Find the first term and the common difference.	The third term of an arithmetic sequence is 8. The eleventh term of the same arithmetic sequence is 40. Find the first term and the common difference. $a = 0, d = -4$

3.2) Arithmetic series

Chapter CONTENTS

Worked example	Your turn
Find the sum of the first 50 terms the sequence which begins: $7, 10, 13, 16, \ldots$	Find the sum of the first 80 terms the sequence which begins: $2, 6, 10, 14, \ldots$ 12800

Worked example	Your turn
Find the sum of the first 50 terms the sequence which begins: $10, 7, 4, 1, -2, \ldots$	Find the sum of the first 80 terms the sequence which begins: $18, 14, 10, 6, 2, \ldots$ -11200

Worked example	Your turn
Find the sum of the first 50 terms the sequence which begins: $p, 3p, 5p, 7p, 9p, \ldots$	Find the sum of the first 80 terms the sequence which begins: $k, 4k, 7k, 10k, 13k, \ldots$ $9560k$

Worked example	Your turn
Find the least number of terms for the sum of $6 + 11 + 16 + \cdots$ to exceed 2000.	Find the least number of terms for the sum of $4 + 9 + 14 + \cdots$ to exceed 2000. 28

Worked example

A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £$(P + 900)$

Salary increases by £(T) each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £P

Salary increases by £$2T$ each year, forming an arithmetic sequence.

For the 10-year period, the total earned is the same for both salary schemes.

a) Find the value of T

b) For this value of T, the salary in Year 10 under Scheme 1 is £25890. Find the value of P

Your turn

A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P

Salary increases by £$(2T)$ each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £$(P + 1800)$

Salary increases by £T each year, forming an arithmetic sequence.

For the 10-year period, the total earned is the same for both salary schemes.

a) Find the value of T

b) For this value of T, the salary in Year 10 under Scheme 2 is £29850. Find the value of P

a) $T = 400$

b) $P = 24450$

Worked example	Your turn
Prove that the sum of the first 200 natural numbers is 20100	Prove that the sum of the first 100 natural numbers is 5050 Proof

Worked example	Your turn
Find the sum of the first 50 odd numbers	Find the sum of the first 50 even numbers 2550

Worked example

Prove that the sum of the first n even numbers is $n^2 + n$

Your turn

Prove that the sum of the first n odd numbers is n^2

Proof

Worked example	Your turn
An arithmetic series is given by $(k+1)+(2k+5)+(3k+9)+\cdots+217$ Given that the sum of the series is 2250, find the value of k	An arithmetic series is given by $(k+1)+(2k+3)+(3k+5)+\cdots+303$ Given that the sum of the series is 2568, find the value of k $k=17$

Worked example	Your turn
The common difference of an arithmetic sequence is 4. The sum of the first 60 terms of this sequence is 7380. Find the first term.	The common difference of an arithmetic sequence is 5. The sum of the first 40 terms of this sequence is 4020. Find the first term. 3

Worked example	Your turn
The common difference of an arithmetic sequence is 4. The sum of the first 60 terms of this sequence is -240. Find the first term.	The common difference of an arithmetic sequence is 5. The sum of the first 40 terms of this sequence is 2620. Find the first term. -32

Worked example	Your turn
The first term of an arithmetic sequence is 3. The sum of the first 50 terms is 2600. Work out the common difference of the sequence.	The first term of an arithmetic sequence is 4. The sum of the first 40 terms is 2500. Work out the common difference of the sequence. 3

Worked example	Your turn
The eighth term of an arithmetic sequence is 11. The fifth term of the same arithmetic sequence is 2. Find the sum of the first 50 terms of this arithmetic sequence.	The tenth term of an arithmetic sequence is 35. The fourth term of the same arithmetic sequence is 17. Find the sum of the first 50 terms of this arithmetic sequence. 4075

3.3) Geometric sequences

Chapter CONTENTS

Worked example	Your turn
Identify the common ratio:	Identify the common ratio:
$1, 3, 9, 27, 81, 243, \ldots$	$1, 2, 4, 8, 16, 32, \ldots$
	$r = 2$
$8, 12, 18, 27, \ldots$	$27, 18, 12, 8, \ldots$
	$r = \frac{2}{3}$
$80, 20, 5, 1.25, \ldots$	$10, 5, 2.5, 1.25, \ldots$
	$r = \frac{1}{2}$
$4, -4, 4, -4, 4, -4, \ldots$	$5, -5, 5, -5, 5, -5, \ldots$
	$r = -1$
$x, -4x^3, 16x^5, \ldots$	$x, -2x^2, 4x^3, \ldots$
	$r = -2x$
$p^4, p^3, p^2, p, \ldots$	$1, p, p^2, p^3, \ldots$
	$r = p$
$4, -2, 1, -0.5, \ldots$	$4, -1, 0.25, -0.0625, \ldots$
	$r = -\frac{1}{4}$

Worked example	Your turn
Find the n^{th} term of the sequence: $3, 6, 12, 24, 48, \ldots$	Find the n^{th} term of the sequence: $2, 6, 18, 54, 162, \ldots$ $u_n = 2 \times 3^{n-1}$
$4, 20, 100, 500, 2500, \ldots$	

Worked example	Your turn
Find the 10th term of the sequence: $3, 6, 12, 24, 48, \ldots$	Find the 10th term of the sequence: $2, 6, 18, 54, 162, \ldots$ 39366
$4, 20, 100, 500, 2500, \ldots$	

Worked example

Find the 10th and nth term of the sequence:

$80, -20, 5, -1.25, 0.3125, \ldots$

Your turn

Find the 10th and nth term of the sequence:

$40, -20, 10, -5, 2.5, \ldots$

$$u_{10} = -\frac{5}{64}$$

$$u_n = (-1)^{n-1} \times \frac{5}{2^{n-4}}$$

Worked example

The second term of a geometric sequence is 6 and the 4^{th} term is 18. The common ratio is positive. Find the exact values of:

a) The common ratio.
b) The first term.
c) The 20^{th} term.

Your turn

The second term of a geometric sequence is 4 and the 4^{th} term is 8. The common ratio is positive. Find the exact values of:

a) The common ratio.
b) The first term.
c) The 10^{th} term.

a) $r = \sqrt{2}$

b) $a = 2\sqrt{2}$

c) $u_{10} = ar^9 = 64$

Worked example

The numbers $2, x$ and $x + 12$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.

b) The 20^{th} term in the sequence.

Your turn

The numbers $3, x$ and $x + 6$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.

b) The 10^{th} term in the sequence.

a) $x = 6$

b) 1536

Worked example

What is the first term in the geometric progression $2, 6, 18, 54, \ldots$ to exceed 1 million?

Your turn

What is the first term in the geometric progression $3, 6, 12, 24, \ldots$ to exceed 1 million?

$$n = 20\,;u_{20} = 1572864$$

Worked example

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x+4, \qquad 10x-2$$

Given the common ratio is positive, find the common ratio and the first term of the sequence

Your turn

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x+6, \qquad 5x-6$$

Given the common ratio is positive, find the common ratio and the first term of the sequence

$$r = 2$$

$$a = 3$$

Worked example

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x-8, \qquad 10x-2$$

Given the common ratio is negative, find the common ratio and the first term of the sequence

Your turn

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x-9, \qquad 5x-3$$

Given the common ratio is negative, find the common ratio and the first term of the sequence

$$r = -2$$

$$a = 3$$

Worked example

The first three terms of a geometric sequence are:

$16, 144, 1296$

Determine whether 944784 is in the sequence

Your turn

The first three terms of a geometric sequence are:

$4, 36, 324$

Determine whether 2125764 is in the sequence

$n = 7$

N is an integer

2125764 is in the sequence

3.4) Geometric series

Chapter CONTENTS

Worked example	Your turn
Find the sum of the first 10 terms: $2, 6, 18, 54, 162, \ldots$	Find the sum of the first 10 terms: $3, 6, 12, 24, 48, \ldots$ 3069
$4, 20, 100, 500, 2500, \ldots$	

Worked example	Your turn
Find the sum of the first 10 terms: $2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \ldots$	Find the sum of the first 10 terms: $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ $\frac{1023}{128}$
$243, -81, 27, -9, 3, \ldots$	

Worked example	Your turn
Find the least value of n such that the sum of $1 + 3 + 9 + 27 + \cdots$ to n terms would exceed 2 000 000.	Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000. $n = 21$

Worked example

A geometric series has first term a and common ratio r.
The sum of the first two terms of the series is 9.
The sum of the first four terms of the series is 45.
Find the two possible geometric sequences.

Your turn

A geometric series has first term a and common ratio r.
The sum of the first two terms of the series is 8
The sum of the first four terms of the series is 80
Find the two possible geometric sequences

$$u_n = 2 \times 3^{n-1}$$
$$u_n = -4 \times (-3)^{n-1}$$

3.5) Sum to infinity

Chapter CONTENTS

Worked example	Your turn
Comment on the sum of these series up to infinity: $1 + 2 + 4 + 8 + 16 + 32 + \cdots$	Comment on the sum of these series up to infinity: $1 + 3 + 9 + 27 + 81 + 243 + \cdots$ Divergent. The sum tends towards infinity
$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 \ldots$	$1 - 3 + 5 - 7 + 9 - 11 + 13 \ldots$ Divergent. The partial sum alternates between positive and negative but the magnitude gets further from 0
$1 + 0.5 + 0.25 + 0.125 + 0.0625 + \cdots$	$1 + 0.25 + 0.0625 + 0.015625 + 0.00390625 \ldots$ Convergent. The sum tends towards a fixed value, in this case $\frac{4}{3}$

Worked example

Find the sum to infinity of the series:

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$$

$$32-16+8-4+2\ldots$$

$$p+p^3+p^5+p^7+p^9+\cdots$$

$$k+\frac{1}{k}+\frac{1}{k^3}+\frac{1}{k^5}+\frac{1}{k^7}+\cdots$$

Your turn

Find the sum to infinity of the series:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

$$2$$

$$27-9+3-1+\frac{1}{3}\ldots$$

$$\frac{81}{4}$$

$$k+k^2+k^3+k^4+k^5+\cdots$$

$$\frac{k}{1-k}$$

$$p+1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\cdots$$

$$\frac{p^2}{p-1}$$

Worked example

Given that the geometric series
$1 - 4x + 16x^2 - 64x^3 + \cdots$ is convergent:

a) Find the range of possible values of x

b) Find an expression for S_∞

Your turn

Given that the geometric series
$1 - 3x + 9x^2 - 27x^3 + \cdots$ is convergent:

a) Find the range of possible values of x

b) Find an expression for S_∞

a) $-\frac{1}{3} < x < \frac{1}{3}$

b) $S_\infty = \frac{1}{1-3x}$

Worked example

The third term of a geometric series is 1.5 and the eighth term is 0.046875.

a) Show that this series is convergent.

b) Find the sum to infinity of this series.

Your turn

The fourth term of a geometric series is 1.08 and the seventh term is 0.23328.

a) Show that this series is convergent.

b) Find the sum to infinity of this series.

a) $|r| = 0.6 < 1$

b) 12.5

Worked example

For a geometric series with first term a and common ratio r, $S_4 = 12.75$ and $S_\infty = 12.8$.

a) Find the possible values of r.

b) Given that the terms in the series alternate between positive and negative values, find the value of a

Your turn

For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a) Find the possible values of r.

b) Given that all the terms in the series are positive, find the value of a.

a) $r = \pm\frac{1}{2}$

b) 8

3.6) Sigma notation

Chapter CONTENTS

Worked example

Write out the terms in the series:

$$\sum_{r=1}^{10}(3r+5)$$

$$\sum_{n=7}^{13}(2-3n)$$

Your turn

Write out the terms in the series:

$$\sum_{k=4}^{8}(2k+1)$$

$9 + 11 + 13 + 15 + 17 = 65$

Worked example

Find the value of a, d and n:

$$\sum_{r=1}^{10}(3r+5)$$

$$\sum_{n=7}^{13}(2-3n)$$

Your turn

Find the value of a, d and n:

$$\sum_{k=4}^{8}(2k+1)$$

$a = 9, d = 2, n = 5$

Worked example

Write in sigma notation:

$8 + 13 + 18 + 23 + 28 + 33 + 38 + 43 + 48 + 53$

$-11 + -13 + -15 + -17 + -19 + \ -21 + -23$

Your turn

Write in sigma notation:

$6 + 7 + 8 + 9 + 10$

$$\sum_{k=4}^{8} (k + 2)$$

Worked example

Write out the terms in the series:

$$\sum_{r=1}^{10} 5 \times 3^{r-1}$$

$$\sum_{n=7}^{13} 2 \times 5^{n-1}$$

Your turn

Write out the terms in the series:

$$\sum_{k=4}^{8} 3 \times 2^{k-1}$$

$$24 + 48 + 96 + 192 + 384 = 744$$

Worked example

Find the value of a, r and n:

$$\sum_{r=1}^{10} 5 \times 3^{r-1}$$

$$\sum_{n=7}^{13} 2 \times 5^{n-1}$$

Your turn

Find the value of a, r and n:

$$\sum_{k=4}^{8} 3 \times 2^{k-1}$$

$a = 24, r = 2, n = 5$

Worked example

Write in sigma notation:

$$3 + 15 + 75 + 375 + 1875 + 9375 + 46875 + 234375 + 1171876 + 5859375$$

$$320 + 640 + 1280 + 2560 + 5120 + 10240 + 20480$$

Your turn

Write in sigma notation:

$$54 + 162 + 486 + 1458 + 4374$$

$$\sum_{k=4}^{8} 2 \times 3^{k-1}$$

Worked example

Evaluate:

$$\sum_{n=9}^{30}(2+7n)$$

Your turn

Evaluate:

$$\sum_{r=10}^{30}(7+2r)$$

987

Worked example

Given that

$$\sum_{r=1}^{k} 3 \times 2^r = 12282$$

Find the value of k

Your turn

Given that

$$\sum_{r=1}^{k} 2 \times 3^r = 59046$$

Find the value of k

$k = 9$

Worked example

A convergent geometric series is given by

$$1 + 2x + 4x^2 + 8x^3$$

a) Find the range of possible values of x

b) Given that

$$\sum_{r=1}^{\infty} (2x)^{r-1} = 2$$

find the value of x

Your turn

A convergent geometric series is given by

$$1 + 4x + 16x^2 + 64x^3$$

a) Find the range of possible values of x

b) Given that

$$\sum_{r=1}^{\infty} (4x)^{r-1} = 2$$

find the value of x

a) $-\frac{1}{4} < x < \frac{1}{4}$

b) $x = \frac{1}{8}$

3.7) Recurrence relations

Chapter CONTENTS

Worked example

Find the first five terms of the sequence:

$$u_{n+1} = 2u_n + 3, u_1 = 3$$

Your turn

Find the first five terms of the sequence:

$$u_{n+1} = 3u_n - 2, u_1 = 4$$

$4, 10, 28, 82, 244$

Worked example

A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 4$$

$$u_{n+1} = pu_n + q, n \geq 1$$

Given that $u_2 = 5$ and $u_3 = 7$, find the values of p and q

Your turn

A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 5$$

$$u_{n+1} = pu_n + q, n \geq 1$$

Given that $u_2 = 13$ and $u_3 = 37$, find the values of p and q

$$p = 3, q = -2$$

Worked example

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 1$$
$$a_{n+1} = (a_n)^2 - ka_n, n \geq 1$$

where k is a constant.

Given that $a_3 = 1$, find the value of:

$$\sum_{r=1}^{100} a_n$$

Your turn

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 1$$
$$a_{n+1} = ka_n + 5, n \geq 1$$

where a is a positive constant.

Given that $a_3 = 41$, find the value of:

$$\sum_{r=1}^{5} a_n$$

901

Worked example

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 2$$
$$a_{n+1} = (a_n)^2 - 2, n \geq 2$$

where $p > 0$

a) Find a_3

b) Given that $a_2 = 2$, find the value of p

c) Find the sum of the first 100 terms

d) Find a_{199}

Your turn

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = p$$
$$a_{n+1} = (a_n)^2 - 1, n \geq 1$$

where $p > 0$

a) Find a_3

b) Given that $a_2 = 0$, find the value of p

c) Find the sum of the first 100 terms

d) Find a_{199}

a) $p^4 - 2p^2$

b) $p = -1$

c) -100

d) -1

Worked example

For each sequence:

i) State whether the sequence is increasing, decreasing or periodic.

ii) If the sequence is periodic, write down its order.

a) $u_{n+1} = u_n - 3,\ u_1 = 7$

b) $u_{n+1} = (u_n)^3,\quad u_1 = 2$

c) $u_{n+1} = \cos(45n°)$

Your turn

For each sequence:

i) State whether the sequence is increasing, decreasing or periodic.

ii) If the sequence is periodic, write down its order.

a) $u_{n+1} = u_n + 3,\ u_1 = 7$

b) $u_{n+1} = (u_n)^2,\quad u_1 = \frac{1}{2}$

c) $u_{n+1} = \sin(90n°)$

a) Increasing

b) Decreasing

c) Periodic, order 4

3.8) Modelling with series

Chapter CONTENTS

Worked example

Bruce starts a new company. In year 1 his profits will be £10 000. He predicts his profits to increase by £5000 each year, so that his profits in year 2 are modelled to be £15 000, in year 3, £20 000 and so on. He predicts this will continue until he reaches annual profits of £50 000. He then models his annual profits to remain at £50 000.

a) Calculate the profits for Bruce's business in the first 20 years.
b) State one reason why this may not be a suitable model.
c) Bruce's financial advisor says the yearly profits are likely to increase by 2.5% per annum. Using this model, calculate the profits for Bruce's business in the first 20 years.

Your turn

Jenny starts a new company. In year 1 her profits will be £20 000. She predicts her profits to increase by £5000 each year, so that her profits in year 2 are modelled to be £25 000, in year 3, £30 000 and so on. She predicts this will continue until she reaches annual profits of
£100 000. She then models her annual profits to remain at £100 000.

a) Calculate the profits for Jenny's business in the first 20 years.
b) State one reason why this may not be a suitable model.
c) Jenny's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Jenny's business in the first 20 years.

a) £1 320 000
b) It is unlikely that Bruce's profits will increase by exactly the same amount each year.
c) £ 661 319.08

Worked example

A company predicts a yearly profit of £210 000 in the year 2031. The company predicts that the yearly profit will rise each year by 4%.

a) Find the predicted profit in the year 2035
b) Find the first year in which the yearly predicted profit exceeds £300 000
c) Find the total predicted profit for the years 2031 to 2042 inclusive, giving your answer to the nearest pound.

Your turn

A company predicts a yearly profit of £120 000 in the year 2013. The company predicts that the yearly profit will rise each year by 5%.

a) Find the predicted profit in the year 2016
b) Find the first year in which the yearly predicted profit exceeds £200 000
c) Find the total predicted profit for the years 2013 to 2023 inclusive, giving your answer to the nearest pound.

a) £138915
b) 2024
c) £1 704 814