3) The normal distribution

3.1) The normal distribution

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Worked example	Your turn
The diameters of a rivet produced by a particular machine, X mm, is modelled as $X \sim N(10, 0.3^2)$. Find: a) $P(X < 10)$ b) $P(9.4 < X < 10.6)$ c) $P(9.1 < X < 10.9)$ d) $P(X = 9.7)$	The diameters of a rivet produced by a particular machine, X mm, is modelled as $X \sim N(8, 0.2^2)$. Find: a) $P(X > 8)$ b) $P(7.8 < X < 8.2)$ c) $P(X = 7.9)$ a) 0.5 b) 0.68 c) 0

Worked example	Your turn
The diameters of a rivet produced by a particular machine, X mm, is modelled as $X \sim N(10, 0.09)$. Find: a) $P(X > 10.6)$ b) $P(X < 9.1)$	The diameters of a rivet produced by a particular machine, X mm, is modelled as $X \sim N(8, 0.04)$. Find: $P(X > 8.2)$ 0.16

Worked example	Your turn
The mass of a group of animals, M grams, is modelled as $$M \sim N(\mu, 25)$$ If 84% of the animals have a mass less than 50 grams, find μ	The mass of a group of animals, M grams, is modelled as $$M \sim N(\mu, 16)$$ If 97.5% of the animals have a mass more than 70 grams, find μ 78

Worked example	Your turn
The mass of a group of animals, M grams, is modelled as $$M \sim N(\mu, \sigma^2)$$ 84% of the animals have a mass less than 70.9 kg and 97.5% of the animals have a mass less than 76.3 kg. Find the population mean and variance.	The mass of a group of animals, M grams, is modelled as $$M \sim N(\mu, \sigma^2)$$ 84% of the animals have a mass more than 52 kg and 97.5% of the animals have a mass more than 47.5 kg. Find the population mean and variance. $$\mu = 56.5\ kg$$ $$\sigma^2 = 20.25$$

3.2) Finding probabilities for normal distributions

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Worked example	Your turn
IQ is distributed using $X \sim N(100, 15^2)$. Find (a) $P(X < 91)$ (b) $P(X \geq 107)$ (c) $P(80 < X < 90)$ (d) $P(X < 86 \text{ or } X > 112)$	IQ is distributed using $X \sim N(100, 15^2)$. Find (a) $P(X < 109)$ (b) $P(X \geq 93)$ (c) $P(110 < X < 120)$ (d) $P(X < 80 \text{ or } X > 106)$ a) 0.7257 (4 dp) b) 0.6796 (4 dp) c) 0.1613 (4 dp) d) 0.4358 (4 dp)

Worked example	Your turn
IQ is distributed using $X \sim N(100, 15^2)$. Adults scoring at least 131 on an IQ test are eligible to join Mensa. Thirty adults take the test. Find the probability that at least three of them are eligible to join.	IQ is distributed using $X \sim N(100, 15^2)$. Adults scoring more than 140 on an IQ test are classified as genius. Twenty adults take the test. Find the probability that at least two are classified as genius. 0.00266 (3 sf)

3.3) The inverse normal distribution function

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Worked example	Your turn
$X \sim N(30, 4)$ Find, correct to two decimal places, the values of a such that: a. $P(X < a) = 0.7$ b. $P(X > a) = 0.45$ c. $P(24 < X < a) = 0.2$	$X \sim N(20, 9)$ Find, correct to two decimal places, the values of a such that: a) $P(X < a) = 0.75$ b) $P(X > a) = 0.4$ c) $P(16 < X < a) = 0.3$ a) $a = 22.0235$ b) $a = 20.76$ c) $a = 19.17$

Worked example

The IQ of a population is distributed using

$$X \sim N(100, 15^2)$$

a) Determine the IQ corresponding to the top 30% of the population.

b) Determine the interquartile range of IQs.

Your turn

Plates made using a particular manufacturing process have a diameter, D cm, which can be modelled using a normal distribution

$$D \sim N(20, 1.5^2)$$

a) Determine the diameter, x, for which 40% of plates have a diameter greater than x

b) Determine the interquartile range of the plate diameters.

a) $x = 20.38$ cm

b) 2.02 cm (2 dp)

Worked example	Your turn
$X \sim N(70, 8^2)$. Using your calculator, determine: a) a such that $P(X > a) = 0.56$ b) b such that $P(65 < X < b) = 0.3$ c) c such that $P(c < X < 66) = 0.15$ d) the interquartile range of X.	$X \sim N(80, 7^2)$. Using your calculator, determine: a) a such that $P(X > a) = 0.65$ b) b such that $P(75 < X < b) = 0.4$ c) c such that $P(c < X < 76) = 0.2$ d) the interquartile range of X. a) $a = 77.303$ (3 dp) b) $b = 82.463$ (3 dp) c) $c = 70.34$ (2 dp) d) 9.44 (2 dp)

3.4) The standard normal distribution

Worked example	Your turn
Z is the number of standard deviations above the mean. Assume $X \sim N(100, 15^2)$ Find z if $X = 100$ $X = 130$ $X = 62.5$	Z is the number of standard deviations above the mean. Assume $X \sim N(100, 15^2)$ Find Z if $X = 85$ $Z = -1$ $X = 165$ $Z = 4.3333\ldots$

Worked example	Your turn
The random variable $X \sim N(40, 5^2)$. Write in terms of $\Phi(z)$ for some value of z. (a) $P(X \leq 45)$ (b) $P(X > 43)$	The random variable $X \sim N(50, 4^2)$. Write in terms of $\Phi(z)$ for some value of z. (a) $P(X < 53)$ (b) $P(X \geq 55)$ a) $\Phi(0.75)$ b) $1 - \Phi(1.25)$

Worked example	Your turn
If $X \sim N(100, 15^2)$, determine, in terms of Φ: (a) $P(X > 70)$ (b) $P(88 < X < 122.5)$	If $X \sim N(100, 15^2)$, determine, in terms of Φ: (a) $P(X > 115)$ (b) $P(77.5 < X < 112)$ a) $1 - \Phi(1)$ b) $\Phi(0.8) + \Phi(1.5) - 1$

Worked example	Your turn
The systolic blood pressure of an adult population, S mmHg, is modelled as a normal distribution with mean 721 and standard deviation 4. A medical research wants to study adults with blood pressures higher than the 90th percentile. Find the minimum blood pressure for an adult included in her study.	The systolic blood pressure of an adult population, S mmHg, is modelled as a normal distribution with mean 127 and standard deviation 16. A medical researcher wants to study adults with blood pressures higher than the 95th percentile. Find the minimum blood pressure for an adult included in her study. $s = 153$ (3 sf)

Worked example	Your turn
Determine: $P(Z > -1.7)$ $P(Z \leq -1.5)$	Determine: $P(Z > -1.3)$ 0.9032 (4 dp)

Worked example	Your turn
Determine: $P(-1 < Z < 0)$ $P(-1.5 < Z < 0.5)$	Determine: $P(-2 < Z < 1)$ 0.8185 (4 dp)

Worked example	Your turn
Determine a such that: $P(Z > a) = 0.3$	Determine a such that: $P(Z > a) = 0.7$
$P(Z < a) = 0.4$	$a = -0.5244$ (4 dp)

Worked example	Your turn
Determine a such that: $P(-a < Z < a) = 0.4$	Determine a such that: $P(-a < Z < a) = 0.6$
$P(-a < Z < a) = 0.5$	$a = -0.8416$ (4 dp)

Worked example	Your turn
Use the percentage points table to find values of z which correspond to the 10% to 80% interpercentile range.	Use the percentage points table to find values of z which correspond to the 20% to 90% interpercentile range. $-0.8416 < z < 1.2816$

3.5) Finding μ and σ

Worked example	Your turn
$X \sim N(\mu, 4^2)$ Given that $P(X > 30) = 0.1$, find the value of μ.	$X \sim N(\mu, 3^2)$ Given that $P(X > 20) = 0.2$, find the value of μ. $\mu = 17.5\ (3sf)$

Worked example

A machine makes metal sheets with width, X cm, modelled as a normal distribution such that

$$X \sim N(70, \sigma^2)$$

(a) Given that $P(X < 64) = 0.02275$, find the value of σ.

(b) Find the 80th percentile of the widths.

Your turn

A machine makes metal sheets with width, X cm, modelled as a normal distribution such that

$$X \sim N(50, \sigma^2)$$

(a) Given that $P(X < 46) = 0.2119$, find the value of σ.

(b) Find the 90th percentile of the widths.

a) $\sigma = 5$

b) $56.4\ cm\ (1dp)$

Worked example	Your turn
A random variable $$X \sim N(\mu, \sigma^2)$$ Given that $P(X < 13) = 0.1964$ and $P(X > 51) = 0.01$, find the values of μ and σ	A random variable $$X \sim N(\mu, \sigma^2)$$ Given that $P(X < 15) = 0.1469$ and $P(X > 35) = 0.025$, find the values of μ and σ $\sigma = 6.64, \mu = 22.0$ (3 sf)

Worked example	Your turn
The time taken for a journey, X, has a normal distribution with mean 200 minutes and standard deviation d minutes. Given that 30% of the journeys take longer than 230 minutes, find the standard deviation.	The time taken for a journey, X, has a normal distribution with mean 100 minutes and standard deviation d minutes. Given that 15% of the journeys take longer than 115 minutes, find the standard deviation. $d = 14.5$

Worked example	Your turn
The time taken for a journey, X, is normally distributed with mean μ days and standard deviation σ days. 15% of journeys are shorter than 532 days. 2.5% are longer than 682 days. Find the values between which the middle 95% of journeys lie.	The time taken for a journey, X, is normally distributed with mean μ days and standard deviation σ days. 2.5% of journeys are shorter than 235 days. 15% are longer than 286 days. Find the values between which the middle 68% of journeys lie. 251 and 285 (3 sf)

Worked example	Your turn
The mass of an animal is found to be normally distributed with mean μ and standard deviation σ. 10% of the animals have a mass less than 9 kg. 5% of the animals have a mass greater than 60 kg. 8 animals are chosen at random. Find the probability that at least two of them have a mass greater than 50 kg.	The mass of an animal is found to be normally distributed with mean μ and standard deviation σ. 5% of the animals have a mass less than 18 kg. 10% of the animals have a mass greater than 30 kg. 9 animals are chosen at random. Find the probability that at least three of them have a mass greater than 25 kg. 0.8832 (4 dp)

3.6) Approximating a binomial distribution

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Worked example	Your turn
A biased coin has $P(tails) = 0.47$. The coin is tossed 200 times and the number of tails is recorded. a) Write a binomial model for X b) Show that X can be approximated with a normal distribution $Y \sim N(\mu, \sigma^2)$ and find the values of μ and σ	A biased coin has $P(tails) = 0.53$. The coin is tossed 100 times and the number of tails is recorded. a) Write a binomial model for X b) Show that X can be approximated with a normal distribution $Y \sim N(\mu, \sigma^2)$ and find the values of μ and σ a) $X \sim (100,0.53)$ b) $Y \sim N(53, 4.99^2)$

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(X = 5)$	$P(X = 3)$
	$\approx P(2.5 \leq Y \leq 3.5)$
$P(X = 4)$	

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(X \leq 5)$	$P(X \leq 3)$ $\approx P(Y \leq 3.5)$
$P(X \leq 4)$	

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n,p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n,p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(X < 5)$	$P(X < 3)$
	$\approx P(Y \leq 2.5)$
$P(X < 4)$	

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(X \geq 5)$	$P(X \geq 3)$
	$\approx P(Y \geq 2.5)$
$P(X \geq 4)$	

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(X > 5)$	$P(X > 3)$
	$\approx P(Y \geq 3.5)$
$P(X > 4)$	

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n, p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(5 \leq X \leq 8)$	$P(3 < X < 6)$
	$\approx P(3.5 \leq Y \leq 5.5)$
$P(4 < X < 7)$	

Worked example	Your turn
Convert these discrete probabilities, where $X \sim B(n,p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$	Convert these discrete probabilities, where $X \sim B(n,p)$ to continuous probabilities, where $Y \sim N(np, np(1-p))$
$P(5 < X \leq 8)$	$P(3 < X \leq 6)$ $\approx P(3.5 \leq Y \leq 6.5)$
$P(4 \leq X < 7)$	

Worked example

For a particular type of flower bulbs, 44% will produce red flowers. A random sample of 160 bulbs is planted.

(a) Calculate the actual probability that there are exactly 70 red flowers.

(b) Use a normal approximation to find a estimate that there are exactly 70 red flowers.

(c) Hence determine the percentage error of the normal approximation for 70 red flowers.

Your turn

For a particular type of flower bulbs, 55% will produce yellow flowers. A random sample of 80 bulbs is planted.

(a) Calculate the actual probability that there are exactly 50 yellow flowers.

(b) Use a normal approximation to find a estimate that there are exactly 50 yellow flowers.

(c) Hence determine the percentage error of the normal approximation for 50 yellow flowers.

a) 0.0365

b) $0.0362\ (4dp)$

c) 0.82% (2 dp)

Worked example	Your turn
The random variable $X \sim B(100, 0.51)$ Use a suitable approximation to estimate: $P(X = 45)$ $P(X > 54)$ $P(X \leq 43)$ $P(47 < X \leq 51)$	The random variable $X \sim B(200, 0.47)$ Use a suitable approximation to estimate: $P(X = 87)$ 0.0346 (4 dp) $P(X > 102)$ 0.1142 (4 dp) $P(X \leq 91)$ 0.3616 (4 dp) $P(89 \leq X < 98)$ 0.4721 (4 dp)

3.7) Hypothesis testing with the normal distribution

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Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given:	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given:
Sample size $n = 40$ Sample mean $\bar{x} = 49$ Population standard deviation $\sigma = 4$ 5% significance level	Sample size $n = 30$ Sample mean $\bar{x} = 49$ Population standard deviation $\sigma = 4$ 10% significance level
H_0: $\mu = 50$ H_1: $\mu < 50$	H_0: $\mu = 50$ H_1: $\mu < 50$ Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{30})$ $\bar{X} \sim N(50, \left(\frac{4}{\sqrt{30}}\right)^2)$ Reject H_0 if $P(\bar{X} \leq 49) < 0.1$ $P(\bar{X} \leq 49) = 0.0854 \ldots < 0.1$ The result is significant. Sufficient evidence to reject H_0. Sufficient evidence to suggest $\mu < 50$

Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given: Sample size $n = 30$ Sample mean $\bar{x} = 51$ Population standard deviation $\sigma = 4$ 10% significance level H_0: $\mu = 50$ H_1: $\mu > 50$	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given: Sample size $n = 40$ Sample mean $\bar{x} = 51$ Population standard deviation $\sigma = 4$ 5% significance level H_0: $\mu = 50$ H_1: $\mu > 50$ Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{40})$ $\bar{X} \sim N(50, \left(\frac{4}{\sqrt{40}}\right)^2)$ Reject H_0 if $P(\bar{X} \geq 51) < 0.05$ $P(\bar{X} \geq 51) = 0.0569 \ldots > 0.05$ The result is not significant. Insufficient evidence to reject H_0. Insufficient evidence to suggest $\mu > 50$

Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given: Sample size $n = 61$ Sample mean $\bar{x} = 51$ Population standard deviation $\sigma = 4$ 10% significance level H_0: $\mu = 50$ H_1: $\mu \neq 50$	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given: Sample size $n = 62$ Sample mean $\bar{x} = 51$ Population standard deviation $\sigma = 4$ 5% significance level H_0: $\mu = 50$ H_1: $\mu \neq 50$ Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{62})$ $\bar{X} \sim N(50, \left(\frac{4}{\sqrt{62}}\right)^2)$ Reject H_0 if $P(\bar{X} \geq 51) < 0.025$ $P(\bar{X} \geq 51) = 0.0245 \ldots < 0.025$ The result is significant. Sufficient evidence to reject H_0. Sufficient evidence to suggest $\mu \neq 50$

Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given: Sample size $n = 62$ Sample mean $\bar{x} = 49$ Population standard deviation $\sigma = 4$ 2% significance level H_0: $\mu = 50$ H_1: $\mu \neq 50$	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Conduct a hypothesis test on the population mean, given: Sample size $n = 61$ Sample mean $\bar{x} = 49$ Population standard deviation $\sigma = 4$ 1% significance level H_0: $\mu = 50$ H_1: $\mu \neq 50$ Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{61})$ $\bar{X} \sim N(50, \left(\frac{4}{\sqrt{61}}\right)^2)$ Reject H_0 if $P(\bar{X} \leq 49) < 0.005$ $P(\bar{X} \leq 49) = 0.0254 \ldots > 0.005$ The result is not significant. Insufficient evidence to reject H_0. Insufficient evidence to suggest $\mu \neq 50$

Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given: Sample size $n = 30$ Population standard deviation $\sigma = 4$ 5% significance level $H_0\colon\ \mu = 50$ $H_1\colon\ \mu < 50$	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given: Sample size $n = 40$ Population standard deviation $\sigma = 4$ 10% significance level $H_0\colon\ \mu = 50$ $H_1\colon\ \mu < 50$ Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{40})$ $\bar{X} \sim N\left(50, \left(\frac{4}{\sqrt{40}}\right)^2\right)$ $Z = \dfrac{\bar{X} - 50}{\frac{4}{\sqrt{40}}}$ $P(Z < z) = 0.1$ -> $z = -1.28155\ldots$ $-1.28155\ldots = \dfrac{\bar{X} - 50}{\frac{4}{\sqrt{40}}}$ $\bar{X} = -1.28155\ldots \times \dfrac{4}{\sqrt{40}} + 50 = 49.189\ldots$ Critical region is $\bar{X} < 49.19$ (2 dp)

Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. a) Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given: Sample size $n = 30$ Population standard deviation $\sigma = 4$ 5% significance level H_0: $\mu = 50$ H_1: $\mu < 50$ b) The mean of the sample of 50 was found to be 48.9. Comment on this observation in light of the critical region.	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. a) Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given: Sample size $n = 40$ Population standard deviation $\sigma = 4$ 10% significance level H_0: $\mu = 50$ H_1: $\mu < 50$ b) The mean of the sample of 50 was found to be 48.9. Comment on this observation in light of the critical region. Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{40})$ $\bar{X} \sim N(50, \left(\frac{4}{\sqrt{40}}\right)^2)$ $Z = \frac{\bar{X} - 50}{\frac{4}{\sqrt{40}}}$ $P(Z < z) = 0.1$ -> $z = -1.28155 \ldots$ $-1.28155 \ldots = \frac{\bar{X} - 50}{\frac{4}{\sqrt{40}}}$ $\bar{X} = -1.28155 \ldots \times \frac{4}{\sqrt{40}} + 50 = 49.189 \ldots$ Critical region is $\bar{X} < 49.19$ (2 dp) b) 48.9 is in the critical region. The result is significant. Sufficient evidence to reject H_0. Sufficient evidence to suggest $\mu < 50$

Worked example

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 40$
Population standard deviation $\sigma = 4$
10% significance level

H_0: $\mu = 50$
H_1: $\mu > 50$

Your turn

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 30$
Population standard deviation $\sigma = 4$
5% significance level

H_0: $\mu = 50$
H_1: $\mu > 50$

Assume H_0 true: $X \sim N(50, 4^2)$

$$\bar{X} \sim N\left(50, \frac{4^2}{30}\right)$$

$$\bar{X} \sim N\left(50, \left(\frac{4}{\sqrt{30}}\right)^2\right)$$

$$Z = \frac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$$

$P(Z > z) = 0.05$ -> $z = 1.64485 \ldots$

$$1.64485 \ldots = \frac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$$

$$\bar{X} = 1.64485 \ldots \times \frac{4}{\sqrt{30}} + 50 = 51.201 \ldots$$

Critical region is $\bar{X} > 51.20$ (2 dp)

Worked example

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2.

a) Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 40$
Population standard deviation $\sigma = 4$
10% significance level

H_0: $\mu = 50$
H_1: $\mu > 50$

b) The mean of the sample of 50 was found to be 50.9. Comment on this observation in light of the critical region.

Your turn

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2.

a) Find the critical region for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 30$
Population standard deviation $\sigma = 4$
5% significance level

H_0: $\mu = 50$
H_1: $\mu > 50$

b) The mean of the sample of 50 was found to be 50.9. Comment on this observation in light of the critical region.

Assume H_0 true: $X \sim N(50, 4^2)$

$$\bar{X} \sim N\left(50, \frac{4^2}{30}\right)$$

$$\bar{X} \sim N\left(50, \left(\frac{4}{\sqrt{30}}\right)^2\right)$$

$$Z = \frac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$$

$P(Z > z) = 0.05$ -> $z = 1.64485\ldots$

$$1.64485\ldots = \frac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$$

$$\bar{X} = 1.64485\ldots \times \frac{4}{\sqrt{30}} + 50 = 51.201\ldots$$

Critical region is $\bar{X} > 51.20$ (2 dp)

b) 50.9 is not in the critical region.
The result is not significant.
Insufficient evidence to reject H_0.
Insufficient evidence to suggest $\mu > 50$

Worked example	Your turn
A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Find the critical regions for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:	A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2. Find the critical regions for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:
Sample size $n = 40$ Population standard deviation $\sigma = 4$ 1% significance level	Sample size $n = 30$ Population standard deviation $\sigma = 4$ 5% significance level
$H_0\colon \mu = 50$ $H_1\colon \mu \neq 50$	$H_0\colon \mu = 50$ $H_1\colon \mu \neq 50$
	Assume H_0 true: $X \sim N(50, 4^2)$ $\bar{X} \sim N(50, \frac{4^2}{30})$ $\bar{X} \sim N\left(50, \left(\frac{4}{\sqrt{30}}\right)^2\right)$ $Z = \dfrac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$ $z = \pm 1.95996\ldots$ $-1.95996\ldots = \dfrac{\bar{X}-50}{\frac{4}{\sqrt{30}}}$ $1.95996\ldots = \dfrac{\bar{X}-50}{\frac{4}{\sqrt{30}}}$ $\bar{X} = -1.95996\ldots \times \frac{4}{\sqrt{30}} + 50 = 48.568\ldots$ $\bar{X} = 1.95996\ldots \times \frac{4}{\sqrt{30}} + 50 = 51.431\ldots.$ Critical region is $\bar{X} < 48.57$ or $\bar{X} > 51.43$ (2 dp)

Worked example

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2.

a) Find the critical regions for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 40$
Population standard deviation $\sigma = 4$
1% significance level

H_0: $\mu = 50$
H_1: $\mu \neq 50$

b) The mean of the sample of 50 was found to be 48.38. Comment on this observation in light of the critical region.

Your turn

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2.

a) Find the critical regions for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 30$
Population standard deviation $\sigma = 4$
5% significance level

H_0: $\mu = 50$
H_1: $\mu \neq 50$

b) The mean of the sample of 50 was found to be 48.38. Comment on this observation in light of the critical region.

a) Assume H_0 true: $X \sim N(50, 4^2)$

$\bar{X} \sim N(50, \frac{4^2}{30})$

$\bar{X} \sim N(50, (\frac{4}{\sqrt{30}})^2)$

$Z = \frac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$

$z = \pm 1.95996 \ldots$

$-1.95996 \ldots = \frac{\bar{X}-50}{\frac{4}{\sqrt{30}}}$ $\quad$ $1.95996 \ldots = \frac{\bar{X}-50}{\frac{4}{\sqrt{30}}}$

$\bar{X} = -1.95996 \ldots \times \frac{4}{\sqrt{30}} + 50$ $\quad$ $\bar{X} = 1.95996 \ldots \times \frac{4}{\sqrt{30}} + 50$

$= 48.568 \ldots$ $\quad$ $= 51.431 \ldots$.

Critical region is $\bar{X} < 48.57$ or $\bar{X} > 51.43$ (2 dp)

b) 48.38 is in the critical region.
The result is significant.
Sufficient evidence to reject H_0.
Sufficient evidence to suggest $\mu \neq 50$

Worked example

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2.

a) Find the critical regions for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 40$

Population standard deviation $\sigma = 4$

1% significance level

H_0: $\mu = 50$

H_1: $\mu \neq 50$

b) The mean of the sample of 50 was found to be 51.64. Comment on this observation in light of the critical region.

Your turn

A random sample of size n is taken from a population X having a normal distribution with mean μ and variance σ^2.

a) Find the critical regions for the test statistic $\bar{X}$ in a hypothesis test on the population mean, given:

Sample size $n = 30$

Population standard deviation $\sigma = 4$

5% significance level

H_0: $\mu = 50$

H_1: $\mu \neq 50$

b) The mean of the sample of 50 was found to be 51.64. Comment on this observation in light of the critical region.

a) Assume H_0 true: $X \sim N(50, 4^2)$

$\bar{X} \sim N(50, \frac{4^2}{30})$

$\bar{X} \sim N(50, (\frac{4}{\sqrt{30}})^2)$

$$Z = \frac{\bar{X} - 50}{\frac{4}{\sqrt{30}}}$$

$z = \pm 1.95996 \ldots$

$-1.95996 \ldots = \frac{\bar{X}-50}{\frac{4}{\sqrt{30}}}$ $\quad$ $1.95996 \ldots = \frac{\bar{X}-50}{\frac{4}{\sqrt{30}}}$

$\bar{X} = -1.95996 \ldots \times \frac{4}{\sqrt{30}} + 50$ $\quad$ $\bar{X} = 1.95996 \ldots \times \frac{4}{\sqrt{30}} + 50$

$= 48.568 \ldots$ $\quad$ $= 51.431 \ldots$.

Critical region is $\bar{X} < 48.57$ or $\bar{X} > 51.43$ (2 dp)

b) 51.64 is in the critical region.

The result is significant.

Sufficient evidence to reject H_0.

Sufficient evidence to suggest $\mu \neq 50$

Worked example

A certain company sells fruit juice in cartons.
The amount of juice in a carton has a normal distribution with a standard deviation of 5ml.
The company claims that the mean amount of juice per carton, μ, is 40ml.
A trading inspector has received complaints that the company is overstating the mean amount of juice per carton and wishes to investigate this complaint. The trading inspector takes a random sample of 36 cartons and finds that the mean amount of juice per carton is 38.9ml.

Using a 5% level of significance, and stating your hypotheses clearly, test whether or not there is evidence to uphold this complaint.

Your turn

A certain company sells fruit juice in cartons.
The amount of juice in a carton has a normal distribution with a standard deviation of 3ml.
The company claims that the mean amount of juice per carton, μ, is 60ml.
A trading inspector has received complaints that the company is overstating the mean amount of juice per carton and wishes to investigate this complaint. The trading inspector takes a random sample of 16 cartons and finds that the mean amount of juice per carton is 59.1ml.

Using a 5% level of significance, and stating your hypotheses clearly, test whether or not there is evidence to uphold this complaint.

X = amount of juice per carton
H_0: $\mu = 60$
H_1: $\mu < 60$
Assume H_0 true: $X \sim N(60, 3^2)$

$$\bar{X} \sim N(60, \frac{3^2}{16})$$

$$\bar{X} \sim N(50, \left(\frac{3}{4}\right)^2)$$

Reject H_0 if $P(\bar{X} \leq 59.1) < 0.05$
$P(\bar{X} \geq 59.1) = 0.1151 \ldots \nless 0.05$
The result is not significant.
Insufficient evidence to reject H_0.
Insufficient evidence to suggest the company is overstating the mean amount of juice per carton.

Worked example

A machine products bolts of diameter D where D has a normal distribution with mean 0.580 cm and standard deviation 0.015 cm. The machine is serviced and after the service a random sample of 50 bolts from the next production run is taken to see if the mean diameter of the bolts has changed from 0.580 cm. The distribution of the diameters of bolts after the service is still normal with a standard deviation of 0.015 cm.

(a) Find, at the 1% level, the critical region for this test, stating your hypotheses clearly.

The mean diameter of the sample of 50 bolts is calculated to be 0.587 cm.

(b) Comment on this observation in light of the critical region.

Your turn

A machine products bolts of diameter D where D has a normal distribution with mean 0.580 cm and standard deviation 0.015 cm. The machine is serviced and after the service a random sample of 50 bolts from the next production run is taken to see if the mean diameter of the bolts has changed from 0.580 cm. The distribution of the diameters of bolts after the service is still normal with a standard deviation of 0.015 cm.

(a) Find, at the 1% level, the critical region for this test, stating your hypotheses clearly.

The mean diameter of the sample of 50 bolts is calculated to be 0.587 cm.

(b) Comment on this observation in light of the critical region.

a) $H_0: \mu = 0.580$

$H_1: \mu \neq 0.580$

Assume H_0 true: $D \sim N(0.580, 0.015^2)$

$$\bar{D} \sim N\left(0.0580, \frac{0.015^2}{50}\right)$$

$$\bar{D} \sim N\left(0.580, \left(\frac{0.015}{\sqrt{50}}\right)^2\right)$$

$$Z = \frac{\bar{D} - 0.580}{\frac{0.015}{\sqrt{50}}}$$

$z = \pm 2.5758 \ldots$

$$-2.5758 \ldots = \frac{\bar{D} - 0.580}{\frac{0.015}{\sqrt{50}}} \qquad 2.5758 \ldots = \frac{\bar{D} - 0.580}{\frac{0.015}{\sqrt{50}}}$$

$$\bar{D} = -2.5758 \ldots \times \frac{0.015}{\sqrt{50}} + 0.580 \qquad \bar{D} = 2.5758 \ldots \times \frac{0.015}{\sqrt{50}} + 0.580$$

Critical region is $\bar{D} < 0.575$ or $\bar{D} > 0.585$ $(3sf)$

b) 0.587 is in the critical region.

The result is significant.

Sufficient evidence to reject H_0.

Sufficient evidence to suggest the mean diameter of bolts has changed.

3.x) Conditional probabilities

Chapter CONTENTS

Worked example	Your turn
The time taken, X minutes, for a flight has a normal distribution with mean μ minutes. Given that $P(X < \mu - 25) = 0.45$, Find $P(X > \mu + 25 \mid X > \mu - 25)$	The time taken, X minutes, for a flight has a normal distribution with mean μ minutes. Given that $P(X < \mu - 15) = 0.35$, Find $P(X > \mu + 15 \mid X > \mu - 15)$ $\frac{7}{13}$

Worked example	Your turn
The length of time, L hours, that a phone will work before it needs charging is normally distributed with a mean of 20 hours and a standard deviation of 3 hours. A person is about to go on a 2 hour journey. Given that it is 25 hours since they last charged their phone, find the probability that their phone will not need charging before the journey is completed.	The length of time, L hours, that a phone will work before it needs charging is normally distributed with a mean of 100 hours and a standard deviation of 15 hours. A person is about to go on a 6 hour journey. Given that it is 127 hours since they last charged their phone, find the probability that their phone will not need charging before the journey is completed. 0.39 (2 sf)