# 2) Functions and graphs

# 2.1) The modulus function

Chapter CONTENTS

| Worked example | Your turn |
|---|---|
| If $f(x) = \|4x + 5\| - 6$, find: | If $f(x) = \|2x - 3\| + 1$, find: |
| a) $f(5)$ | a) $f(5)$ |
| | $8$ |
| b) $f(-2)$ | b) $f(-2)$ |
| | $-6$ |
| c) $f(1)$ | c) $f(1)$ |
| | $2$ |

| Worked example | Your turn |
|---|---|
| Sketch: | Sketch: |

## Worked example

Sketch:

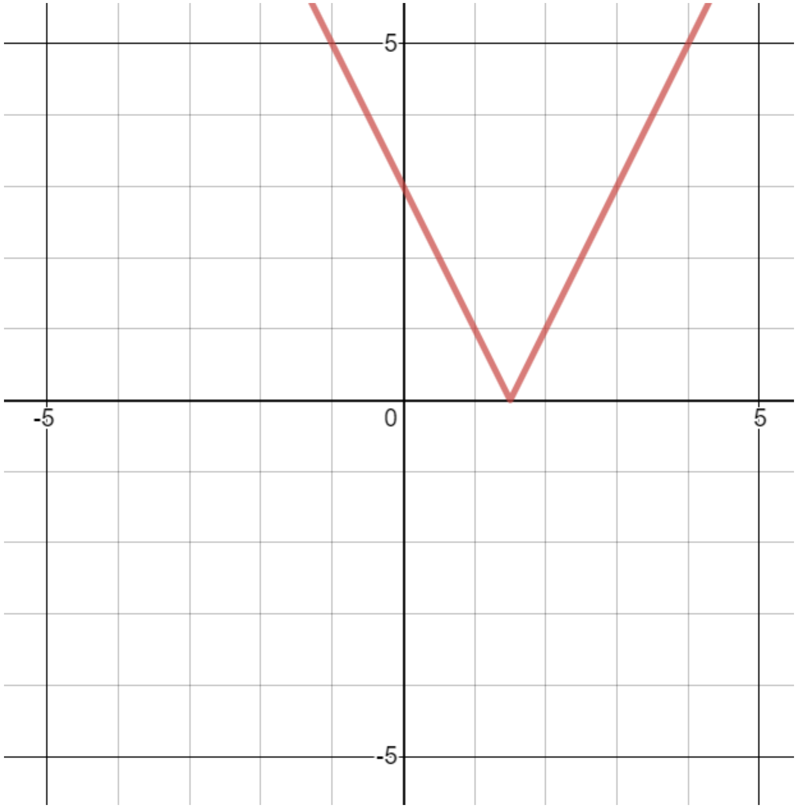
$$y = |3x - 2|$$

$$y = |2 - 3x|$$

## Your turn

Sketch:

$$y = |2x - 3|$$

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| Worked example | Your turn |
| --- | --- |
| Solve: $$\lvert 3x-2\rvert = 7$$ $$\lvert 2-3x\rvert = 6$$ | Solve: $$\lvert 2x-3\rvert = 5$$ $$x=-1, x=4$$ |

| Worked example | Your turn |
|---|---|
| Solve: | Solve: |

**Worked example**

Solve:

$$|5x-2| = 3-\frac{1}{3}x$$

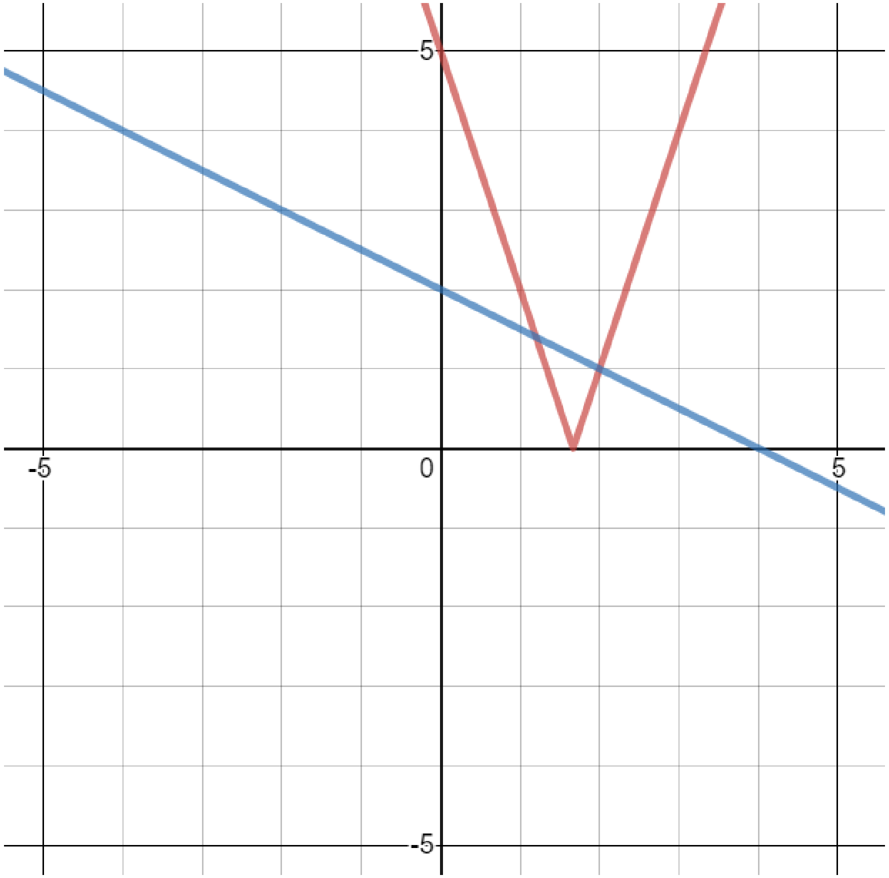
$$|5-3x| = \frac{1}{2}x+2$$

**Your turn**

Solve:

$$|3x-5| = 2-\frac{1}{2}x$$

$$x=\frac{6}{5}, x=2$$

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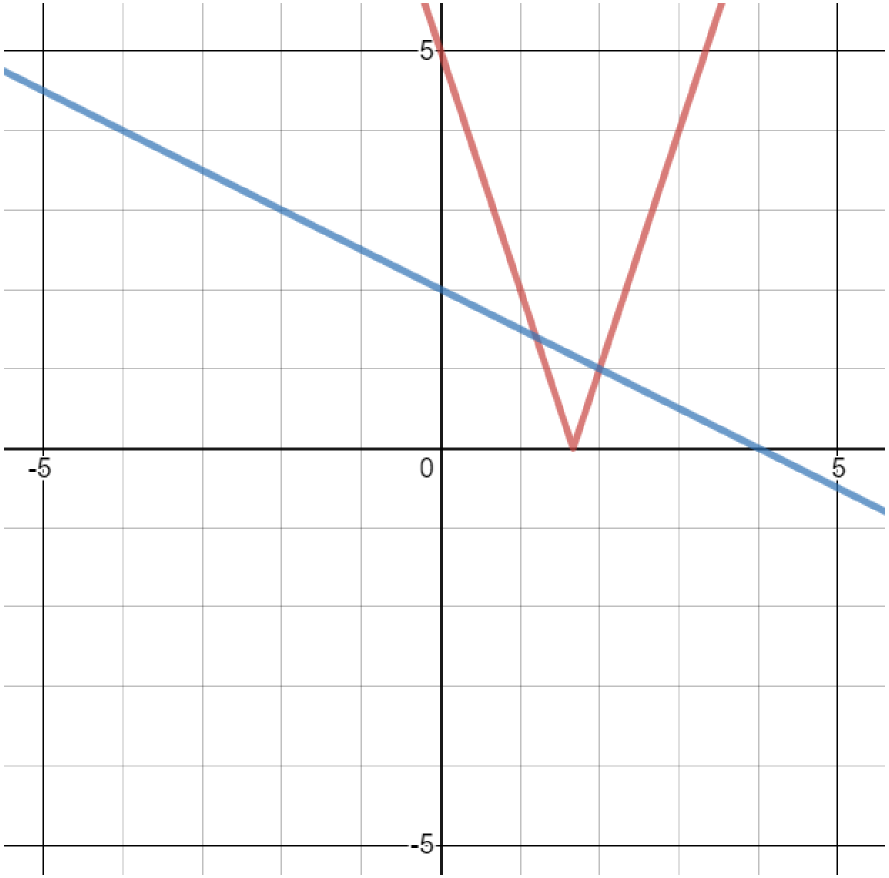
| Worked example | Your turn |
| --- | --- |
| Solve: | Solve: |
| $$\lvert 5x-2\rvert < 3-\frac{1}{3}x$$ | $$\lvert 3x-5\rvert < 2-\frac{1}{2}x$$ |
| $$\lvert 5-3x\rvert \le \frac{1}{2}x+2$$ | $$\frac{6}{5} < x < 2$$ |

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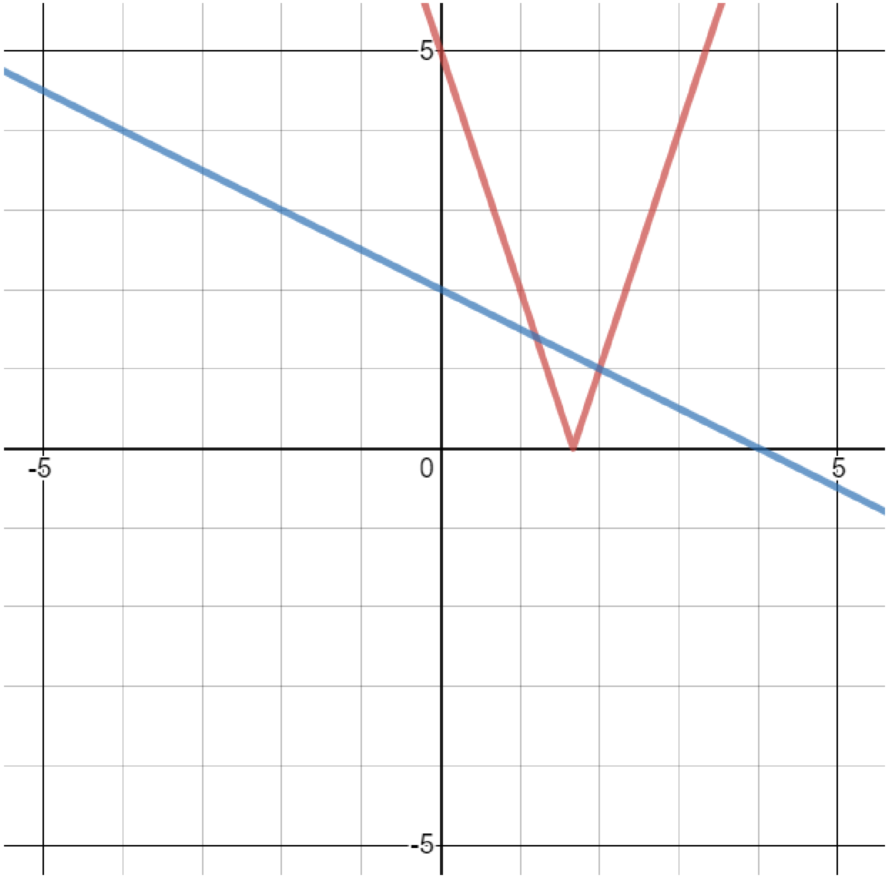
| Worked example | Your turn |
|---|---|
| Solve: | Solve: |
| $$\lvert 5x-2\rvert > 3-\frac{1}{3}x$$ | $$\lvert 3x-5\rvert > 2-\frac{1}{2}x$$ |
| $$\lvert 5-3x\rvert \geq \frac{1}{2}x+2$$ | $$x<\frac{6}{5}\cup x>2$$ |

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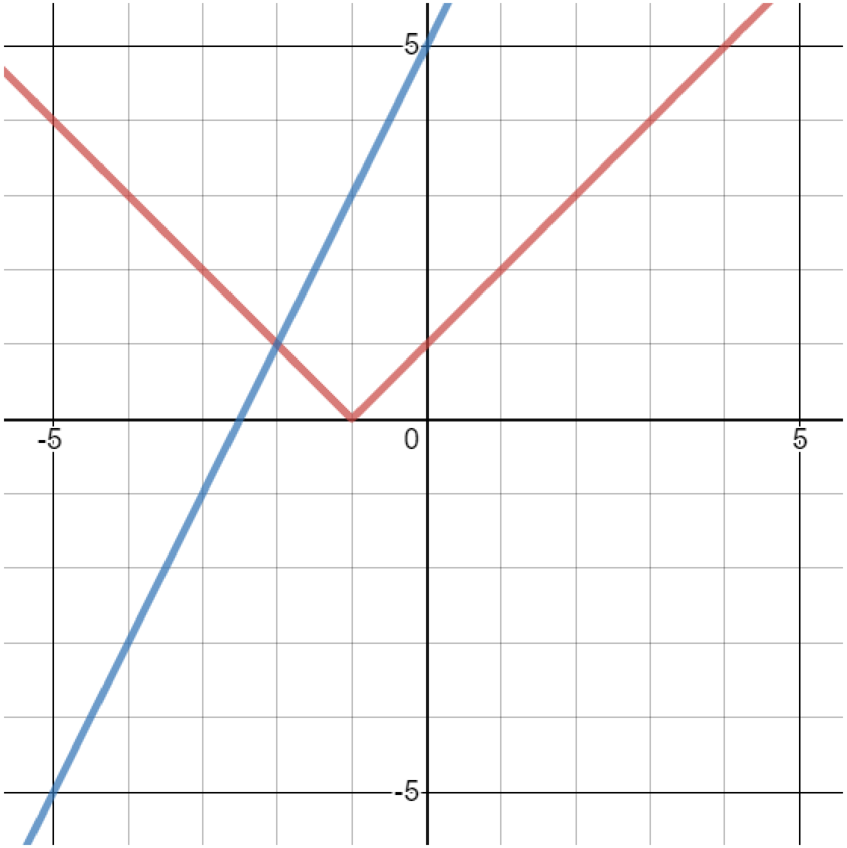
| Worked example | Your turn |
| --- | --- |
| Solve: $$\|x+3\| = 5x+2$$ $$\|x+3\| = x+2$$ | Solve: $$\|x+1\| = 2x+5$$ $$x = -2$$   |

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# 2.2) Functions and mappings

| Worked example | Your turn |
|---|---|
| State whether:<br>• the mapping is one-to-one, many-to-one, or one-to-many<br>• the mapping is a function | State whether:<br>• the mapping is one-to-one, many-to-one, or one-to-many<br>• the mapping is a function |
| $f(x) = 2x - 3, \qquad x \in \mathbb{R}$ | $p(x) = x^3, \qquad x \in \mathbb{R}$<br>One-to-one: a function |
| $g(x) = x^2, \qquad x \in \mathbb{R}$ | $q(x) = \left\lvert \frac{1}{x} \right\rvert, \qquad x \in \mathbb{R}$<br>Many-to-one: Not a function |
| $h(x) = \frac{1}{x}, \qquad x \in \mathbb{R}$ | $r(x) = \sqrt{x}, x \in \mathbb{R}, x \geq 0, \qquad x \in \mathbb{R}$<br>One-to-one: a function |
| $i(x) = \sqrt{x}, \qquad x \in \mathbb{R}$ | $s(x) = \pm\sqrt{x}, x \in \mathbb{R}, x \geq 0$<br>One-to-many: Not a function |

| Worked example | Your turn |
| --- | --- |
| Write down the largest possible domain for: | Write down the largest possible domain for: |
| $f(x) = \frac{1}{x-3}$ | $p(x) = \frac{6}{x+4}$ <br> $x \neq -4$ |
| $g(x) = \frac{2}{7x-21}$ | $q(x) = \frac{7}{5x+20}$ <br> $x \neq -4$ |
| $h(x) = \frac{3}{2x^2-x-3}$ | $r(x) = \frac{8}{3x^2+10x-8}$ <br> $x \neq \frac{2}{3}, x \neq -4$ |
| $i(x) = \frac{4x+5}{x^2-64}$ | $s(x) = \frac{9x-10}{x^2-16}$ <br> $x \neq -4, x \neq 4$ |

| Worked example | Your turn |
| --- | --- |
| Write down the largest possible domain for: | Write down the largest possible domain for: |
| $f(x) = \sqrt{x-3}$ | $p(x) = \sqrt{x+4}$ |
| | $x \geq -4$ |
| $g(x) = \sqrt{7x-21}$ | $q(x) = \sqrt{5x+20}$ |
| | $x \geq -4$ |
| $h(x) = \sqrt{7x+21}$ | $r(x) = \sqrt{5x-20}$ |
| | $x \geq 4$ |
| $i(x) = \sqrt{21-7x}$ | $s(x) = \sqrt{20-5x}$ |
| | $x \leq 4$ |

## Worked example

Write down the largest possible domain for:

$$f(x) = \frac{\sqrt{x+3}}{x^2 - 2x}$$

$$g(x) = \frac{x^3 - 2x^2}{\sqrt{x^2 + 5x + 6}}$$

## Your turn

Write down the largest possible domain for:

$$h(x) = \frac{\sqrt{x+4}}{x^4 - 25x^2}$$

$$x \geq -4, x \neq 0, x \neq 5$$

# Worked example

Find the range of the following functions:

$f(x) = 2x - 3, \qquad x = \{1, 2, 3, 4\}$

$g(x) = 3 - 2x, \qquad x \in \mathbb{R}, x \leq 0$

$h(x) = 3 - 2x, \qquad x \in \mathbb{R}, 2 < x < 5$

# Your turn

Find the range of the following functions:

$p(x) = 3x - 2, \qquad x = \{1, 2, 3, 4\}$

$p(x) = \{1, 4, 7, 10\}$

$q(x) = 2 - 3x, \qquad x \in \mathbb{R}, x > 0$

$q(x) < 2$

$r(x) = 2 - 3x, \qquad x \in \mathbb{R}, -3 < x \leq 4$

$-10 \leq r(x) < 11$

# Worked example

Find the range of the following functions:

$$f(x) = x^4, \qquad x = \{1, 2, 3, 4\}$$

$$g(x) = x^4, \qquad x \in \mathbb{R}, x \leq 0$$

$$h(x) = x^4, \qquad x \in \mathbb{R}, -2 \leq x < 5$$

# Your turn

Find the range of the following functions:

$$p(x) = x^2, \qquad x = \{1, 2, 3, 4\}$$

$$p(x) = \{1, 4, 9, 16\}$$

$$q(x) = x^2, \qquad x \in \mathbb{R}, x > 0$$

$$q(x) > 0$$

$$r(x) = x^2, \qquad x \in \mathbb{R}, -3 < x \leq 4$$

$$0 \leq r(x) \leq 16$$

| Worked example | Your turn |
|---|---|
| Find the range of the following functions: | Find the range of the following functions: |
| $f(x) = \frac{1}{x}, \qquad x = \{-1, -2, -3, -4\}$ | $p(x) = \frac{1}{x}, \qquad x = \{1, 2, 3, 4\}$ |
| | $p(x) = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$ |
| $g(x) = \frac{1}{x-2}, \qquad x \in \mathbb{R}, x \leq 1$ | $q(x) = \frac{1}{x+2}, \qquad x \in \mathbb{R}, x > -1$ |
| | $q(x) < 1$ |
| $h(x) = \frac{1}{x+3}, \qquad x \in \mathbb{R}, -2 \leq x < 5$ | $r(x) = \frac{1}{x-5}, \qquad x \in \mathbb{R}, -3 < x \leq 4$ |
| | $-1 \leq r(x) < -\frac{1}{8}$ |

# Worked example

Find the range of the following functions:

$$f(x) = \frac{1}{x}, \qquad x \in \mathbb{R}, x \neq 0$$

$$g(x) = \frac{1}{x} + 2, \qquad x \in \mathbb{R}, x \neq 0$$

# Your turn

Find the range of the following functions:

$$h(x) = \frac{1}{x} - 3, \qquad x \in \mathbb{R}, x \neq 0$$

$$h(x) \in \mathbb{R}$$

| Worked example | Your turn |
|---|---|
| Find the range of the following functions: | Find the range of the following functions: |
| $f(x) = e^x + 5, \qquad x \in \mathbb{R}$ | $p(x) = e^x + 8, \qquad x \in \mathbb{R}$ |
| | $p(x) > 8$ |
| $g(x) = e^x - 4, \qquad x \in \mathbb{R}, x > 0$ | $q(x) = e^x - 7, \qquad x \in \mathbb{R}, x < 0$ |
| | $-7 < x < -6$ |
| $h(x) = -e^x - 3, \qquad x \in \mathbb{R}, x \leq 0$ | $r(x) = -e^x - 6, \qquad x \in \mathbb{R}, x \geq 0$ |
| | $r(x) \leq -7$ |

# Worked example

Find the range of the following functions:

$f(x) = \ln x + 5, \qquad x \in \mathbb{R}, x > 0$

$g(x) = \ln x - 4, \qquad x \in \mathbb{R}, x > 0$

# Your turn

Find the range of the following functions:

$h(x) = \ln x + 3, \qquad x \in \mathbb{R}, x > 0$

$h(x) \in \mathbb{R}$

| Worked example | Your turn |
|---|---|
| The function $f$ is defined by<br>$f: x \to x^2 - 8x + 3, \qquad x \in \mathbb{R}, 0 \leq x \leq 5$<br>Find the range of $f$. | The function $h$ is defined by<br>$h: x \to x^2 - 4x + 1, \qquad x \in \mathbb{R}, 0 \leq x < 5$<br>Find the range of $h$.<br>$-3 \leq h(x) < 6$ |
| The function $f$ is defined by<br>$g: x \to x^2 + 6x - 2, \qquad x \in \mathbb{R}, -5 < x \leq 0$<br>Find the range of $f$. | |

| Worked example | Your turn |
| --- | --- |
| The function $f$ is defined by $f(x) = x^2 - 8x + 27$ and has domain $x \geq a$. Given that $f(x)$ is a one-to-one function, find the smallest possible value of the constant $a$ | The function $h$ is defined by $h(x) = x^2 - 6x + 20$ and has domain $x \geq a$. Given that $f(x)$ is a one-to-one function, find the smallest possible value of the constant $a$<br>$a = 3$ |
| The function $g$ is defined by $g(x) = x^2 + 4x + 15$ and has domain $x \leq a$. Given that $g(x)$ is a one-to-one function, find the smallest possible value of the constant $a$ | |

| Worked example | Your turn |
|---|---|
| The function $f(x)$ is defined by | The function $f(x)$ is defined by |

**Worked example**

The function $f(x)$ is defined by

$$f: x \to \begin{cases} 2 - 5x, & x < 1 \\ x^2 - 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.

b) Solve $f(x) = 22$

**Your turn**

The function $f(x)$ is defined by

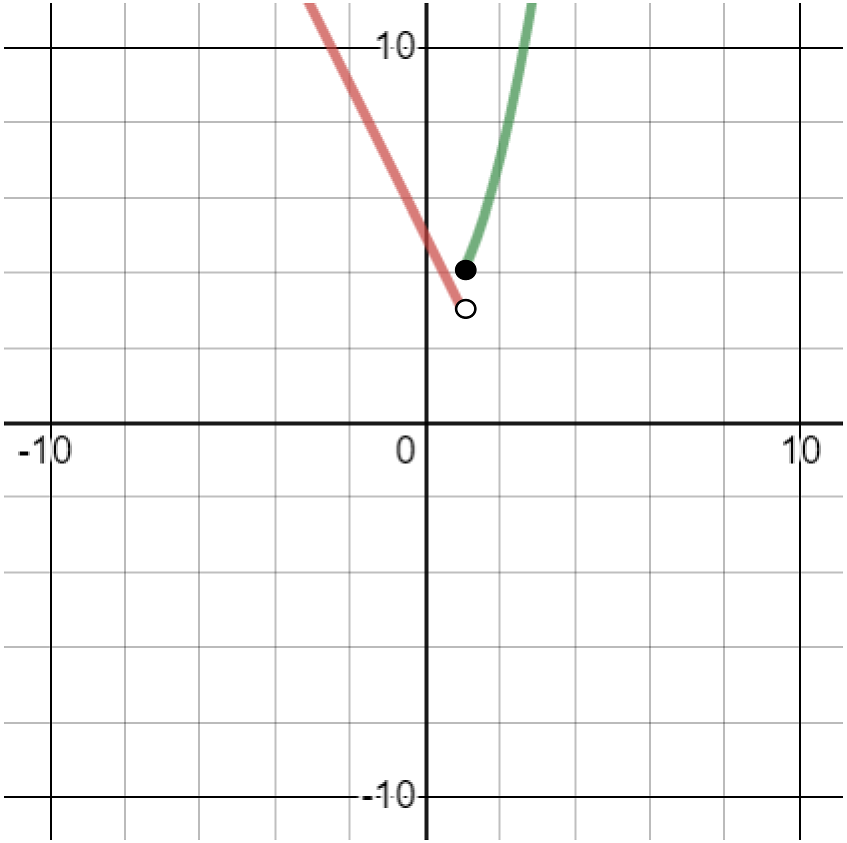
$$f: x \to \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.

b) Solve $f(x) = 19$

a) Sketch ; $f(x) > 3$

b) $x = 4, x = -7$

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| Worked example | Your turn |
| --- | --- |
| Find the inverse function: | Find the inverse function: |
| $$f(x) = \frac{2x+3}{4}$$ | $$h(x) = \frac{4x-3}{2}$$ |
| | $$h^{-1}(x) = \frac{2x+3}{4}$$ |
| $$g(x) = \frac{3x-2}{5}$$ | |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $$f(x) = \frac{x}{2} - 3$$ $$g(x) = 2 + \frac{x}{3}$$ | Find the inverse function: $$h(x) = 5 + \frac{x}{4}$$ $$h^{-1}(x) = 4(x - 5)$$ |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: | Find the inverse function: |
| $$f(x) = \frac{x}{2} - 3$$ | $$h(x) = \frac{x}{5} + 4$$ |
| | $$h^{-1}(x) = 5(x - 4)$$ |
| $$g(x) = \frac{x - 3}{2}$$ | |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $f(x) = 3(x-2)$ | Find the inverse function: $h(x) = 5(x+4)$ |
| | $h^{-1}(x) = \frac{x}{5} - 4$ |
| $g(x) = 2(x+3)$ | |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: | Find the inverse function: |
| $$f(x) = 2 + \frac{3}{x}$$ | $$h(x) = \frac{5}{x} + 4$$ |
| | $$h^{-1}(x) = \frac{5}{x - 4}$$ |
| $$g(x) = \frac{2}{x} - 3$$ | |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $$f(x) = \frac{2}{4x - 3}$$ $$g(x) = \frac{3}{2 - 5x}$$ | Find the inverse function: $$h(x) = \frac{4}{5 - 3x}$$ $$h^{-1}(x) = \frac{5x - 4}{3x}$$ |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $f(x) = 3\sqrt{x}$ <br><br> $g(x) = 5\sqrt[3]{x}$ | Find the inverse function: $h(x) = 4\sqrt{x}$ <br><br> $h^{-1}(x) = \frac{x^2}{16}$ |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $f(x) = 3\sqrt{x} - 2$ <br> $g(x) = 5\sqrt[3]{x} + 3$ | Find the inverse function: $h(x) = 4\sqrt{x} - 5$ <br> $h^{-1}(x) = \frac{(x+5)^2}{16}$ |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: | Find the inverse function: |
| $f(x) = \sqrt{\frac{x-2}{x+3}}$ | $h(x) = \sqrt{\frac{5x-4}{x+3}}$ |
| | $h^{-1}(x) = \frac{3x^2+4}{5-x^2}$ |
| $g(x) = \sqrt[3]{\frac{3x-2}{x-4}}$ | |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $f(x) = x^2 + 4x - 5$ <br><br> $g(x) = x^2 - 6x + 3$ | Find the inverse function: $h(x) = x^2 + 8x - 5$ <br> $h^{-1}(x) = -4 + \sqrt{x + 21}$ |

| Worked example | Your turn |
| --- | --- |
| Find the inverse function: $$f(x) = 2x^2 - 10x + 9$$ $$g(x) = 3x^2 - 8x + 2$$ | Find the inverse function: $$h(x) = 2x^2 - 12x + 3$$ $$h^{-1}(x) = 3 + \sqrt{\frac{x+15}{2}}$$ |

# 2.3) Composite functions

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| Worked example | Your turn |
| --- | --- |
| $f(x) = 3x - 2$, and $g(x) = x^2 - 4$ | $f(x) = 3x + 2$, and $g(x) = x^2 + 4$ |
| Find: | Find: |
| $fg(x)$ | $fg(x)$<br>$fg(x) = 3x^2 + 14$ |
| $gf(x)$ | $gf(x)$<br>$gf(x) = 9x^2 + 12x + 8$ |
| $f^2(x)$ | $f^2(x)$<br>$f^2(x) = 9x + 8$ |
| $g^2(x)$ | $g^2(x)$<br>$g^2(x) = x^4 + 8x^2 + 20$ |

| Worked example | Your turn |
|---|---|
| $f(x) = 3x - 2$, and $g(x) = x^2 - 4$ | $f(x) = 3x + 2$, and $g(x) = x^2 + 4$ |
| Find: | Find: |
| $fg(1)$ | $fg(4)$<br>62 |
| $gf(-2)$ | $gf(-3)$<br>53 |
| $f^2(3)$ | $f^2(2)$<br>26 |
| $g^2(-4)$ | $g^2(-1)$<br>29 |

| Worked example | Your turn |
| --- | --- |
| $f(x) = 3x - 2$, and $g(x) = x^2 - 4$ | $f(x) = 3x + 2$, and $g(x) = x^2 + 4$ |
| Solve: | Find: |
| $fg(a) = 13$ | $fg(a) = 62$ |
| | $a = \pm 4$ |
| $gf(b) = 12$ | $gf(b) = 293$ |
| | $b = 5, b = -\frac{19}{3}$ |

# Worked example

The functions $f$ and $g$ are defined by

$$f: x \to |3x - 12|$$
$$g: x \to \frac{x+2}{3}$$

a) Find $fg(2)$

b) Solve $fg(x) = x$

# Your turn

The functions $f$ and $g$ are defined by

$$f: x \to |2x - 8|$$
$$g: x \to \frac{x+1}{2}$$

a) Find $fg(3)$

b) Solve $fg(x) = x$

a) 4

b) $x = \frac{7}{2}$

## Worked example

The function $g$ is defined by

$$g: x \to 4 - 3x, \qquad x \in \mathbb{R}$$

Solve the equation

$$g^2(x) + [g(x)]^2 = 0$$

## Your turn

The function $g$ is defined by

$$g: x \to 3 - 4x, \qquad x \in \mathbb{R}$$

Solve the equation

$$g^2(x) + [g(x)]^2 = 0$$

$$x = 0, x = \frac{1}{2}$$

# Worked example

The functions $f$ and $g$ are defined by

$$f: x \to e^{x} + 3, \qquad x \in \mathbb{R}$$
$$g: x \to \ln x, \qquad x > 0$$

Find $fg(x)$, giving your answer in its simplest form.

The functions $f$ and $g$ are defined by

$$f: x \to e^{3x} - 2, \qquad x \in \mathbb{R}$$
$$g: x \to 4\ln(x+1), \qquad x > -1$$

Find $fg(x)$, giving your answer in its simplest form.

# Your turn

The functions $f$ and $g$ are defined by

$$f: x \to e^{2x} + 4, \qquad x \in \mathbb{R}$$
$$g: x \to 3\ln(x-1), \qquad x > 1$$

Find $fg(x)$, giving your answer in its simplest form

$$fg(x) = (x-1)^6 + 4$$

| Worked example | Your turn |
|---|---|
| The functions $f$ and $g$ are defined by<br>$f: x \to 2^x + 3, \quad x \in \mathbb{R}$<br>$g: x \to \log_2 x, \quad x > 0$<br>Find $fg(x)$, giving your answer in its simplest form.<br><br>The functions $f$ and $g$ are defined by<br>$f: x \to 3^{2x} - 1, \quad x \in \mathbb{R}$<br>$g: x \to 4\log_3(x+5), \quad x > -5$<br>Find $fg(x)$, giving your answer in its simplest form. | The functions $f$ and $g$ are defined by<br>$f: x \to 2^{3x} + 4, \quad x \in \mathbb{R}$<br>$g: x \to 5\log_2(x-1), \quad x > 1$<br>Find $fg(x)$, giving your answer in its simplest form<br><br>$fg(x) = (x-1)^{15} + 4$ |

| Worked example | Your turn |
|---|---|
| $f(x) = \frac{1}{x-1}, x \neq 1$<br>Find an expression for $f^2(x)$ and $f^3(x)$ | $f(x) = \frac{1}{x+1}, x \neq -1$<br>Find an expression for $f^2(x)$ and $f^3(x)$<br>$f^2(x) = \frac{x+1}{x+2}, x \neq -1, x \neq -2$<br>$f^3(x) = \frac{x+2}{2x+3}, x \neq -1, x \neq -2, x \neq -\frac{3}{2}$ |

| Worked example | Your turn |
| --- | --- |
| A function $f$ has domain $-3 \leq x \leq 12$ and is linear from $(-3, 9)$ to $(0, 6)$ and from $(0, 6)$ to $(12,10)$.<br>Find the value of $f^2(0)$ | A function $f$ has domain $-4 \leq x \leq 13$ and is linear from $(-4, 9)$ to $(0, 5)$ and from $(0, 5)$ to $(13, 31)$.<br>Find the value of $f^2(0)$<br><br>$15$ |

# 2.4) Inverse functions

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| Worked example | Your turn |
|---|---|
| Find the inverse functions: $f(x) = 4x + 3, \quad x \in \mathbb{R}$ <br> $g(x) = 4 - 3x, \quad x \in \mathbb{R}$ | Find the inverse function: $h(x) = 3 - 4x, \quad x \in \mathbb{R}$ <br> $h^{-1}(x) = \frac{3 - x}{4}, \quad x \in \mathbb{R}$ |

## Worked example

Find the inverse functions:

$$f(x) = \frac{x-2}{2x+1}, \qquad x \neq \frac{1}{2}$$

$$g(x) = \frac{2x+3}{4x-5}, \qquad x \neq \frac{5}{4}$$

## Your turn

Find the inverse function:

$$h(x) = \frac{x+2}{2x-1}, \qquad x \neq \frac{1}{2}$$

$$h^{-1}(x) = \frac{x+2}{2x-1}, \qquad x \neq \frac{1}{2}$$

| Worked example | Your turn |
|---|---|
| Find the inverse functions: $f(x) = 3x^2 - 5, \qquad x \geq 0$ <br><br> $g(x) = 4x^2 + 6, \qquad x \geq 0$ | Find the inverse function: $h(x) = 2x^2 - 7, \qquad x \geq 0$ <br><br> $h^{-1}(x) = \sqrt{\frac{x+7}{2}}, \qquad x \geq -7$ |

| Worked example | Your turn |
| --- | --- |
| Find the inverse functions:<br>$f(x) = x^2 + 4x + 3, \qquad x \geq -2$<br><br>$g(x) = x^2 - 8x - 5, \qquad x \geq 5$ | Find the inverse function:<br>$h(x) = x^2 - 6x - 5, \qquad x \geq 3$<br><br>$h^{-1}(x) = 3 + \sqrt{x + 14}, \qquad x \geq -14$ |

| Worked example | Your turn |
|---|---|
| Find the inverse functions:<br>$f(x)=\frac{2}{x-5}, \quad x\in\mathbb{R}, x\neq 5$<br><br>$g(x)=\frac{7}{x+2}, \quad x\in\mathbb{R}, x\neq -2$ | Find the inverse function:<br>$h(x)=\frac{3}{x-1}, \quad x\in\mathbb{R}, x\neq 1$<br><br>$h^{-1}(x)=\frac{3+x}{x}, \quad x\neq 0$ |

## Worked example

Find the inverse functions:

$$f(x) = e^x - 3, \qquad x \in \mathbb{R}$$

$$g(x) = e^x + 4, \qquad x \in \mathbb{R}$$

## Your turn

Find the inverse function:

$$h(x) = e^x - 5, \qquad x \in \mathbb{R}$$

$$h^{-1}(x) = \ln(x + 5), \qquad x > -5$$

| Worked example | Your turn |
| --- | --- |
| Find the inverse functions: | Find the inverse function: |
| $f(x) = \ln x - 3, \qquad x > 0$ | $h(x) = \ln(x-5), \qquad x > 5$ |
| | $h^{-1}(x) = e^x + 5, \qquad x \in \mathbb{R}$ |
| $g(x) = \ln(x-4), \qquad x > 4$ | |

# Worked example

$f(x) = \sqrt{x-3}\ \{x \in \mathbb{R}, x \geq 3\}$

a) State the range of $f(x)$

b) Find the function $f^{-1}(x)$ and state its domain and range

c) Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$

# Your turn

$p(x) = \sqrt{x-2}\ \{x \in \mathbb{R}, x \geq 2\}$

a) State the range of $p(x)$

b) Find the function $p^{-1}(x)$ and state its domain and range

c) Sketch $y = p(x)$, $y = p^{-1}(x)$ and $y = x$

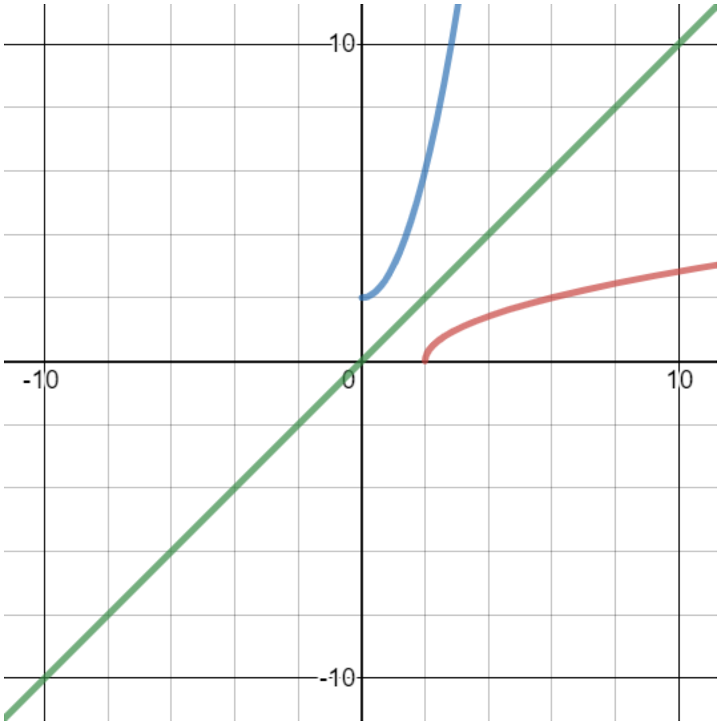
a) $p(x) \geq 0$

b) $p^{-1}(x) = x^2 + 2$

Domain: $x \in \mathbb{R}, x \geq 0$

Range: $p^{-1}(x) \geq 2$

c) Sketch




Graphs used with permission from DESMOS: https://www.desmos.com/

| Worked example | Your turn |
| --- | --- |
| $f(x) = x^2 - 5, x \in \mathbb{R}, x \geq 0.$ | $p(x) = x^2 - 3, x \in \mathbb{R}, x \geq 0.$ |

## Worked example

$f(x) = x^2 - 5, x \in \mathbb{R}, x \geq 0.$

a) State the range of $f(x)$

b) Find the function $f^{-1}(x)$ and state its domain and range

c) Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$

d) Solve the equation $f(x) = f^{-1}(x)$.

## Your turn

$p(x) = x^2 - 3, x \in \mathbb{R}, x \geq 0.$

a) State the range of $p(x)$

b) Find the function $p^{-1}(x)$ and state its domain and range

c) Sketch $y = p(x)$, $y = p^{-1}(x)$ and $y = x$

d) Solve the equation $p(x) = p^{-1}(x)$.

a) $p(x) \geq -3$

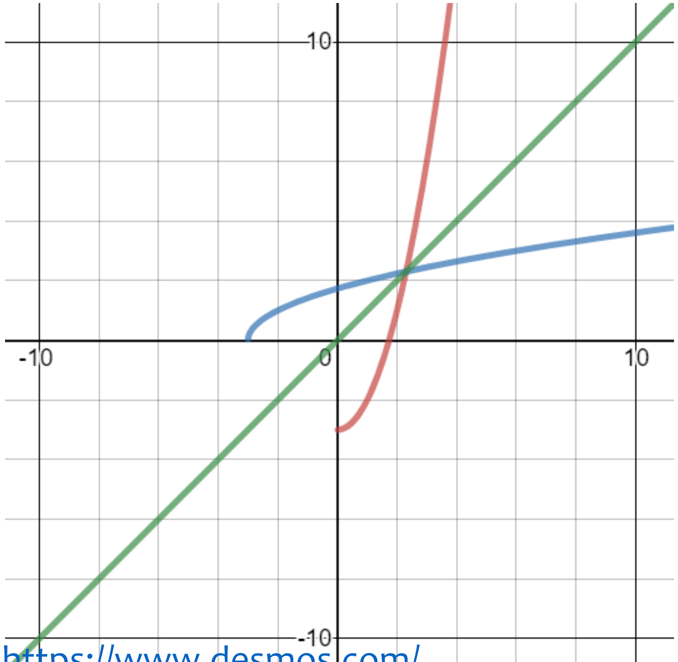
b) $p^{-1}(x) = \sqrt{x+3}$

Domain: $x \in \mathbb{R}, x \geq -3$

Range: $p^{-1}(x) \geq 0$

c) Sketch

d) $x = \frac{1+\sqrt{13}}{2}$




Graphs used with permission from DESMOS: https://www.desmos.com/

# 2.5) $y = |f(x)|$ and $y = f(|x|)$

Chapter CONTENTS

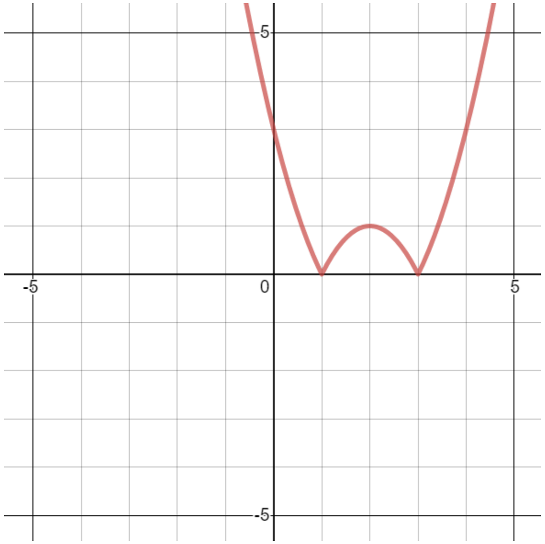
| Worked example | Your turn |
|---|---|
| $f(x) = x^2 + 4x + 3$ | $f(x) = x^2 - 4x + 3$ |

Sketch:

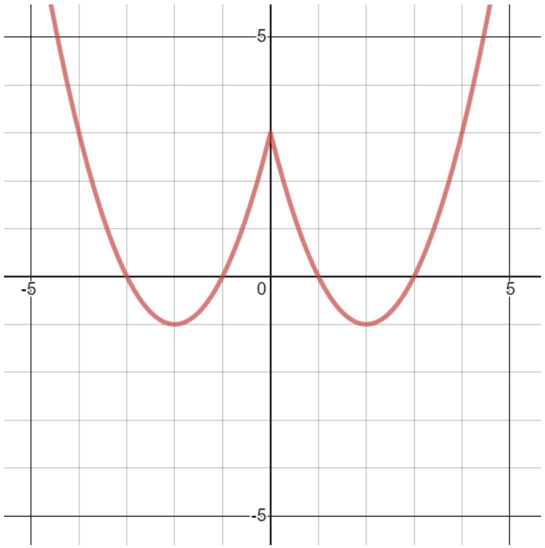
- $y = \|f(x)\|$
- $y = f(\|x\|)$

Sketch:

- $y = |f(x)|$




- $y = f(|x|)$




Graphs used with permission from DESMOS: https://www.desmos.com/

| Worked example | Your turn |
|---|---|
| $f(x) = x^2 + 3x - 10$ | $f(x) = x^2 - 3x - 10$ |

**Worked example**

$f(x) = x^2 + 3x - 10$

Sketch:

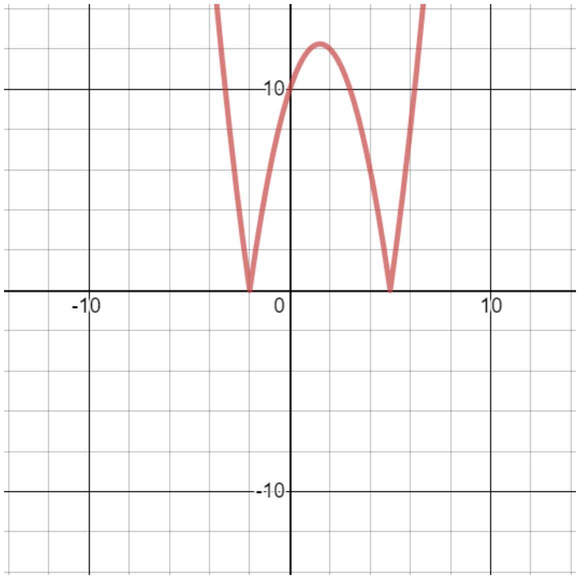
- $y = |f(x)|$
- $y = f(|x|)$

**Your turn**

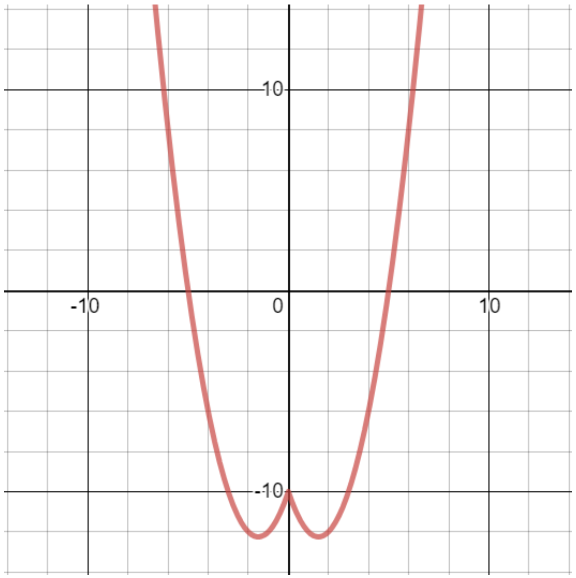
$f(x) = x^2 - 3x - 10$

Sketch:

- $y = |f(x)|$



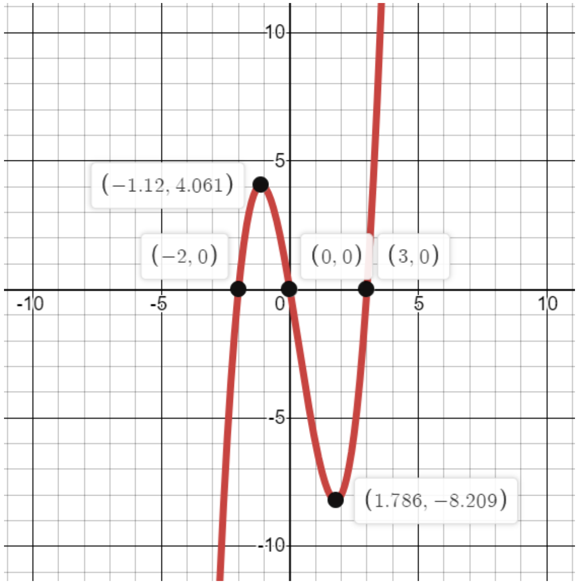

- $y = f(|x|)$




Graphs used with permission from DESMOS: https://www.desmos.com/

# Worked example

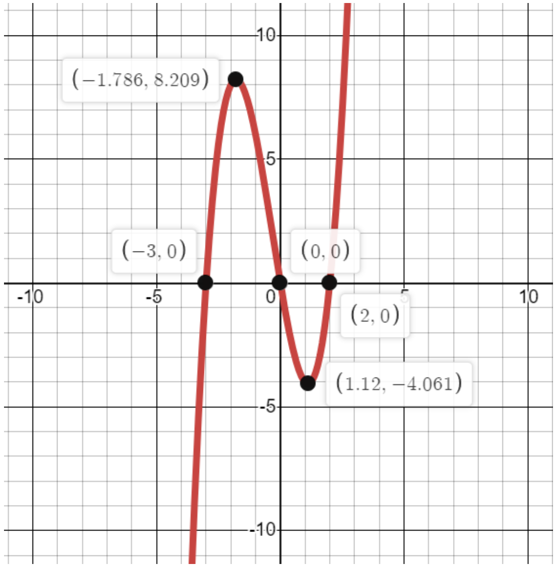
A sketch of $y = f(x)$ is shown.



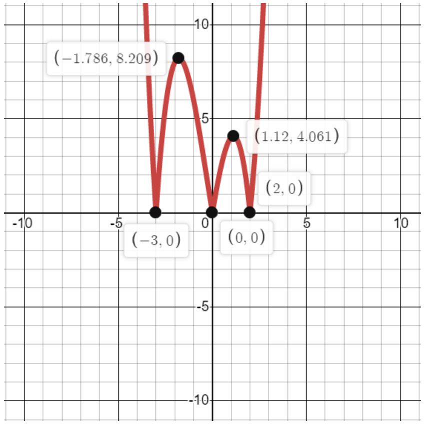

Sketch $y = |f(x)|$ and $y = f(|x|)$ on separate axes.

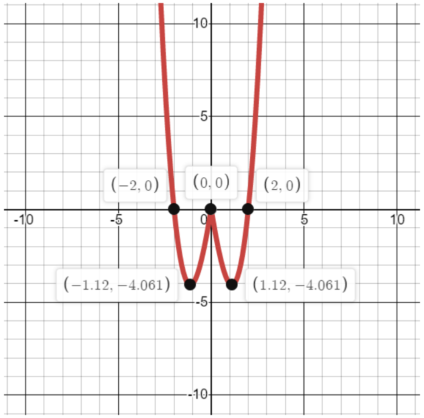
# Your turn

A sketch of $y = f(x)$ is shown.




Sketch $y = |f(x)|$ and $y = f(|x|)$ on separate axes.




Graphs used with permission from DESMOS: https://www.desmos.com/

| # Worked example | # Your turn |
|---|---|
| $y = \cos x\,, \qquad -2\pi \leq x \leq 2\pi$ | $y = \sin x\,, \qquad -2\pi \leq x \leq 2\pi$ |

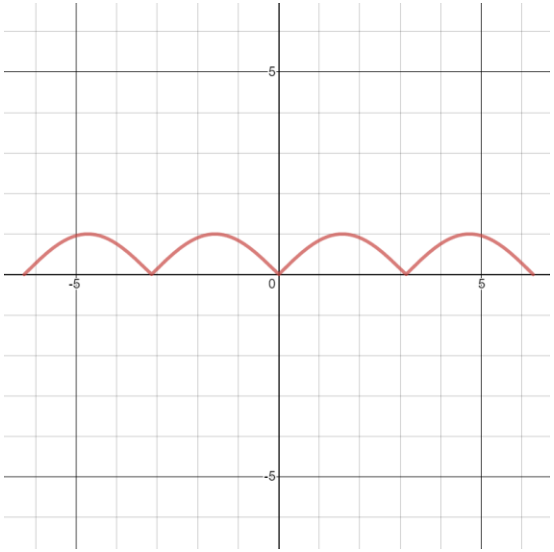
Sketch:

- $y = \lvert\cos x\rvert$

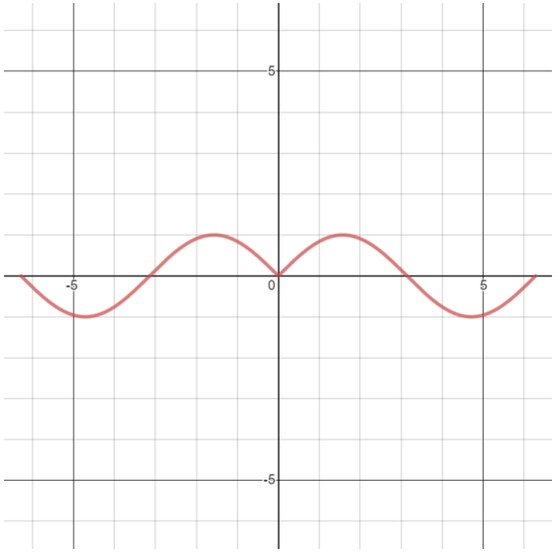
- $y = \cos\lvert x\rvert$

Sketch:

- $y = \lvert\sin x\rvert$




- $y = \sin\lvert x\rvert$

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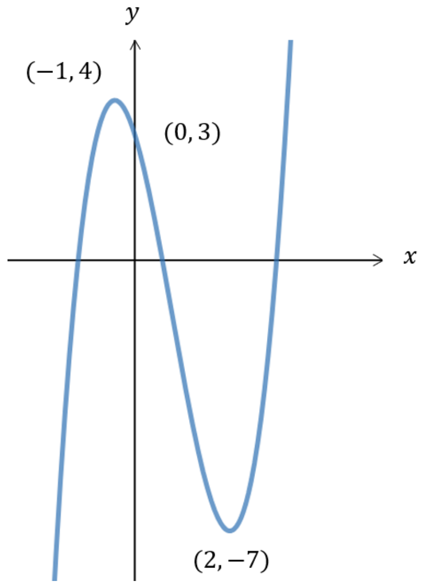
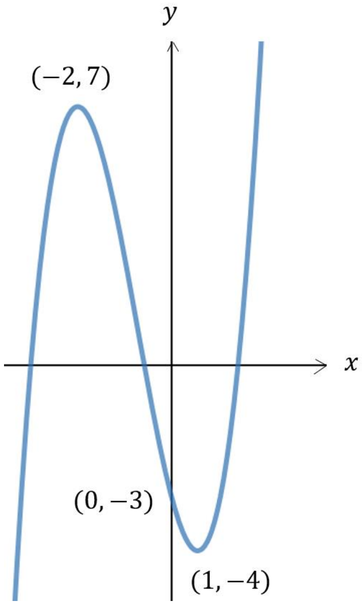
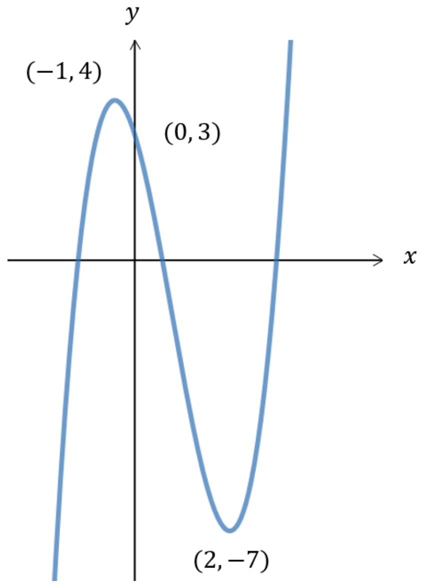
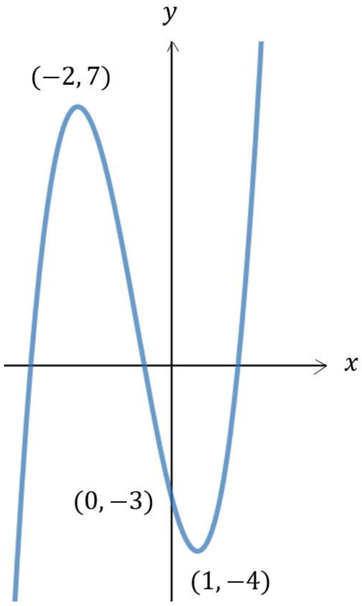
# 2.6) Combining transformations

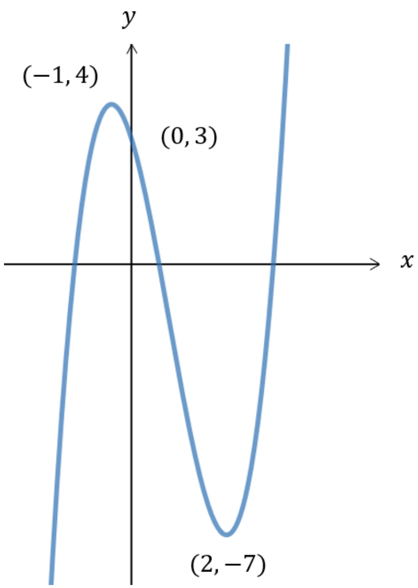
Chapter CONTENTS

| Worked example | Your turn |
| --- | --- |
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |
|  |  |
| Sketch the graph of $y = f(x) - 2$ | Sketch the graph of $y = f(x) + 3$ |
| | Correct sketch<br>New $y$-intercept: $(0, 0)$<br>New turning points: $(-2, 10)$ and $(1, -1)$ |

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |
|  |  |
| Sketch the graph of $y = f(x - 2)$ | Sketch the graph of $y = f(x + 3)$ |
| | Correct sketch<br>New turning points: $(-5, 7)$ and $(-2, -4)$ |

# Worked example

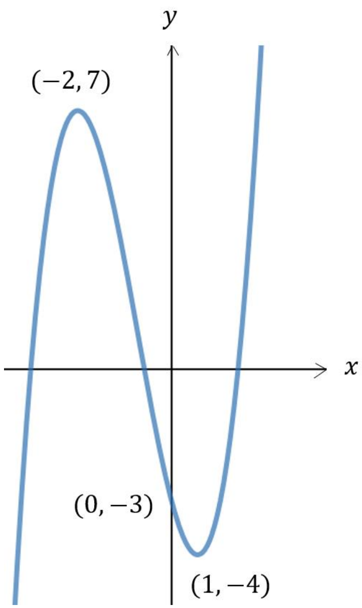
A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = 3f(x)$

# Your turn

A sketch of the graph $y = f(x)$ is shown:

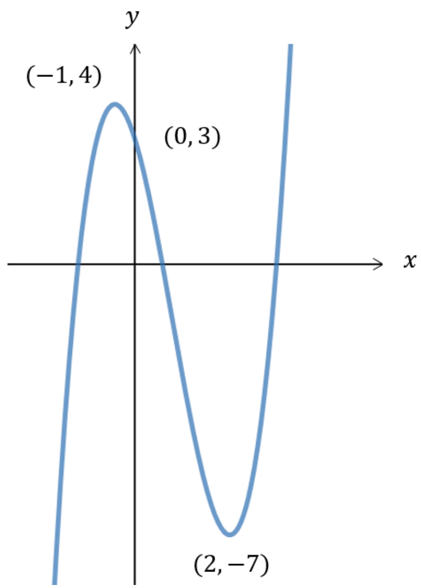



Sketch the graph of $y = 2f(x)$

Correct sketch
New $y$-intercept: $(0, -6)$
New turning points: $(-2, 14)$ and $(1, -8)$

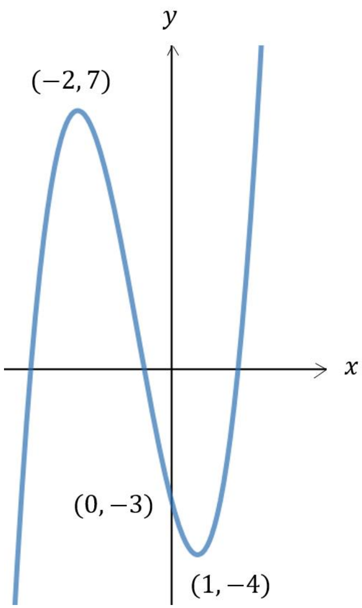
| Worked example | Your turn |
|---|---|

A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = f(2x)$

A sketch of the graph $y = f(x)$ is shown:



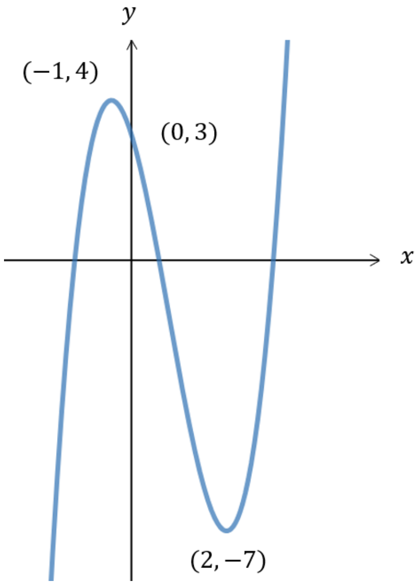

Sketch the graph of $y = f(3x)$

Correct sketch

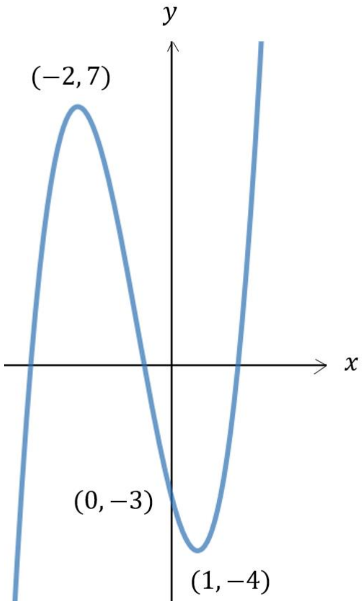
New $y$-intercept: $(0, -3)$

New turning points: $(-\frac{2}{3}, 14)$ and $(\frac{1}{3}, -8)$

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |




Sketch the graph of $y = -f(x)$



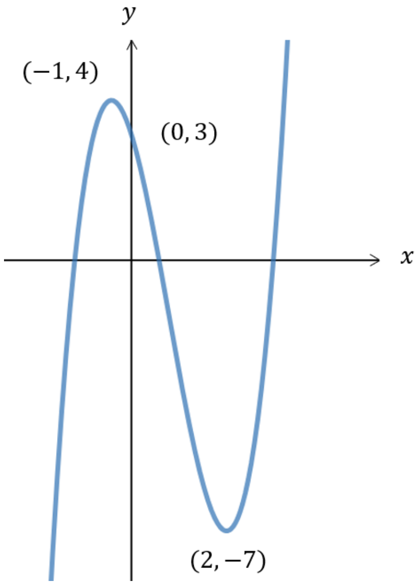

Sketch the graph of $y = -f(x)$

Correct sketch

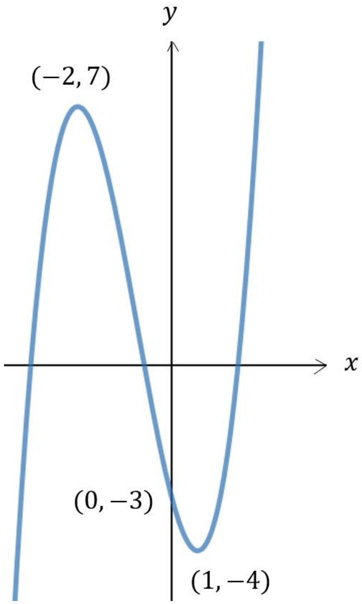
New $y$-intercept: $(0, 3)$

New turning points: $(-2, -7)$ and $(1, 4)$

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |




Sketch the graph of $y = f(-x)$

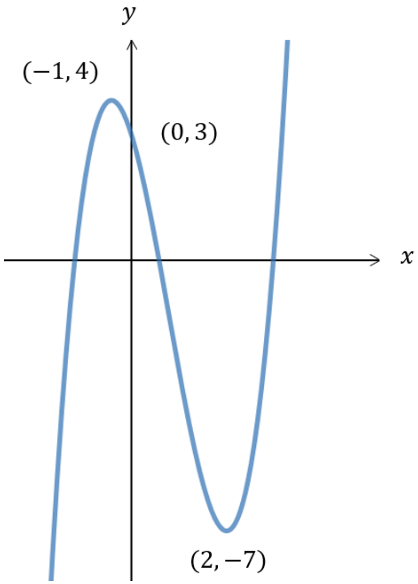
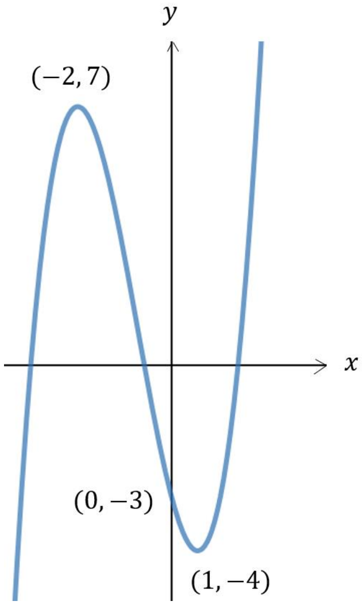



Sketch the graph of $y = f(-x)$

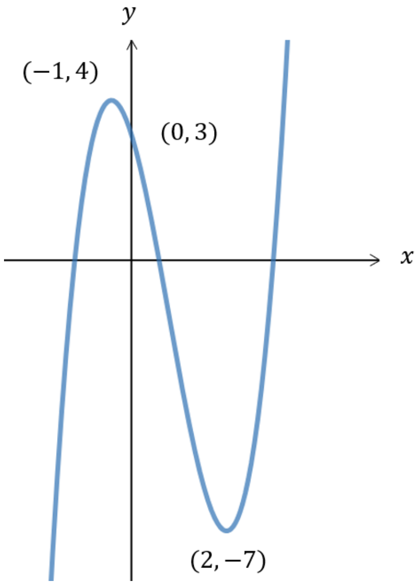
Correct sketch

New $y$-intercept: $(0, -3)$

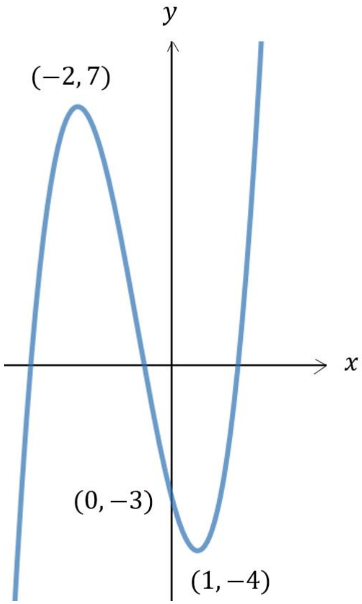
New turning points: $(2, -7)$ and $(-1, 4)$

| Worked example | Your turn |
| --- | --- |
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |
|  |  |
| Sketch the graph of $y = f(x+2)+3$ | Sketch the graph of $y = f(x-3)-2$ |
| | Correct sketch<br>New turning points: $(1, 5)$ and $(4, -6)$ |

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |



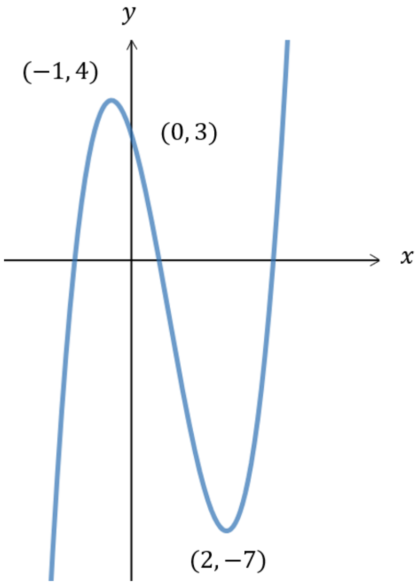

Sketch the graph of $y = -f(x) + 3$



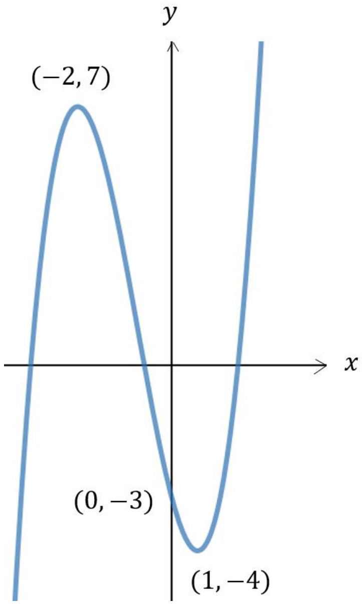

Sketch the graph of $y = -f(x) - 2$

Correct sketch
New $y$-intercept: $(0, 1)$
New turning points: $(-2, -9)$ and $(1, 2)$

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |

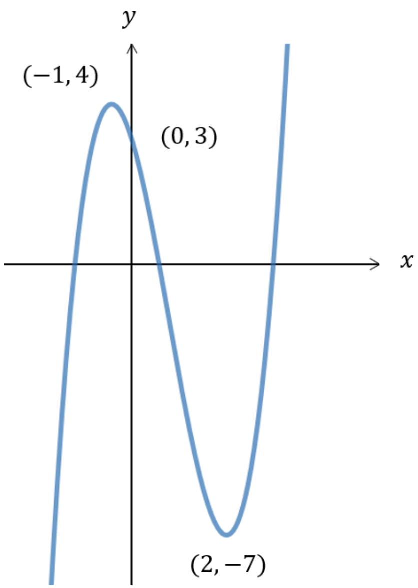
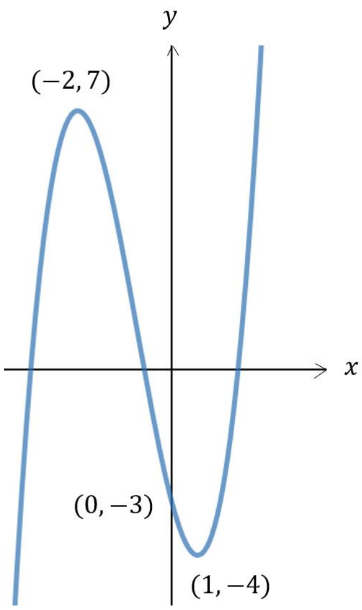


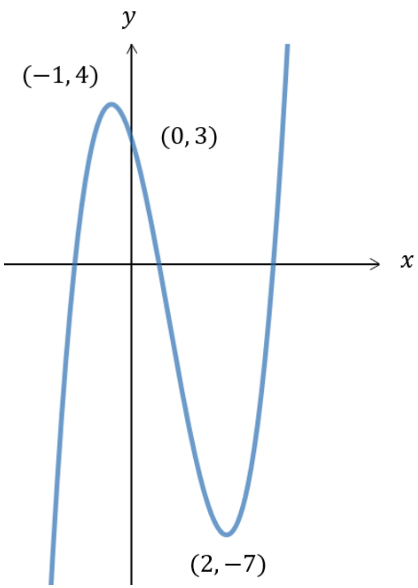
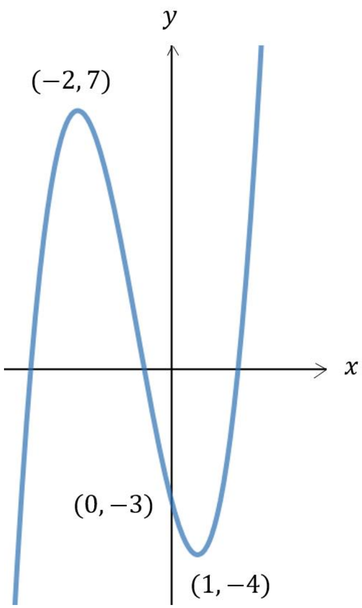

Sketch the graph of $y = f(-x) - 3$




Sketch the graph of $y = f(-x) + 2$

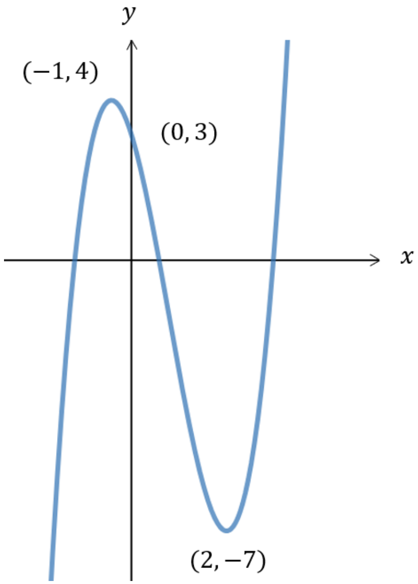
Correct sketch
New $y$-intercept: $(0, -1)$
New turning points: $(2, -5)$ and $(-1, 6)$

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |
|  |  |
| Sketch the graph of $y = 3f(x) + 2$ | Sketch the graph of $y = 2f(x) + 3$ |
| | Correct sketch<br>New $y$-intercept: $(0, -3)$<br>New turning points: $(-2, 17)$ and $(1, -5)$ |

| Worked example | Your turn |
| --- | --- |
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |
|  |  |
| Sketch the graph of $y = f(2x) - 3$ | Sketch the graph of $y = f(3x) - 2$ |
| | Correct sketch<br>New $y$-intercept: $(0, -5)$<br>New turning points: $(-\frac{2}{3}, 5)$ and $(\frac{1}{3}, -6)$ |

# Worked example

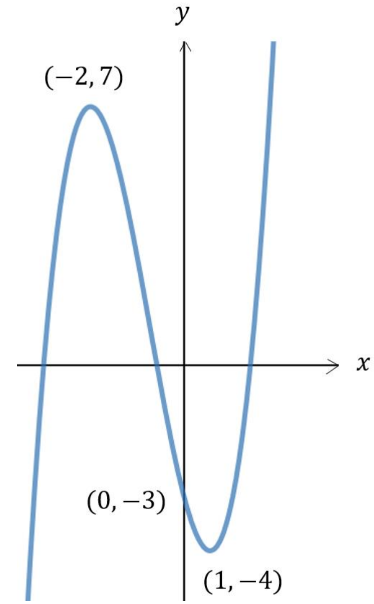
A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = -f(3x) + 2$

# Your turn

A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = -f(2x) - 3$

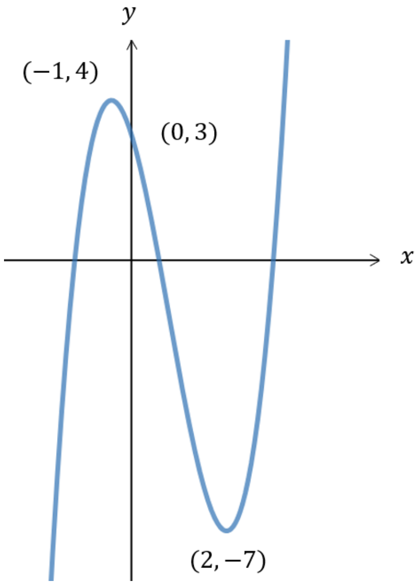
Correct sketch

New $y$-intercept: $(0, 0)$

New turning points: $(-1, -10)$ and $(\frac{1}{2}, 1)$

# Worked example

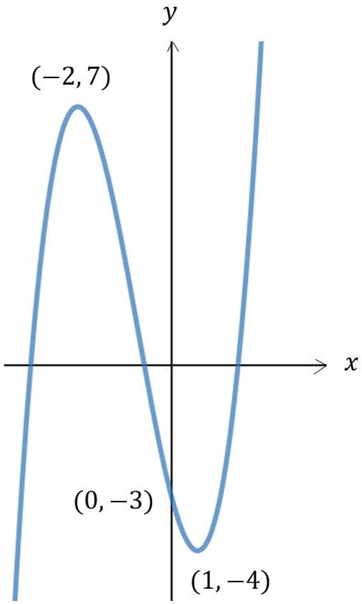
A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = 5f(x-2) - 3$

# Your turn

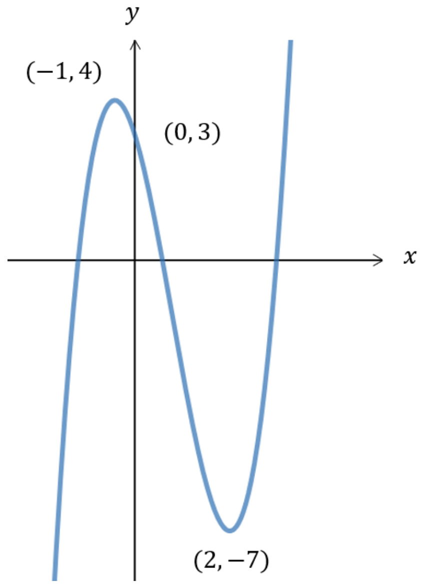
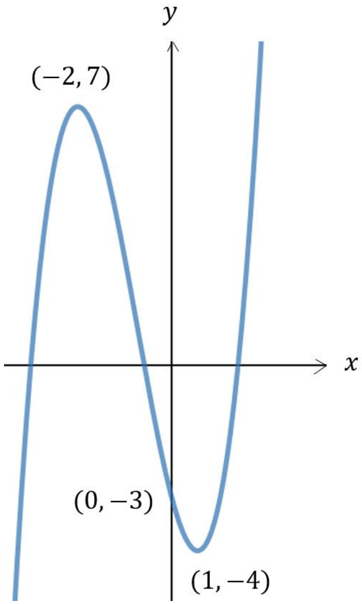
A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = 7f(x+3) + 2$

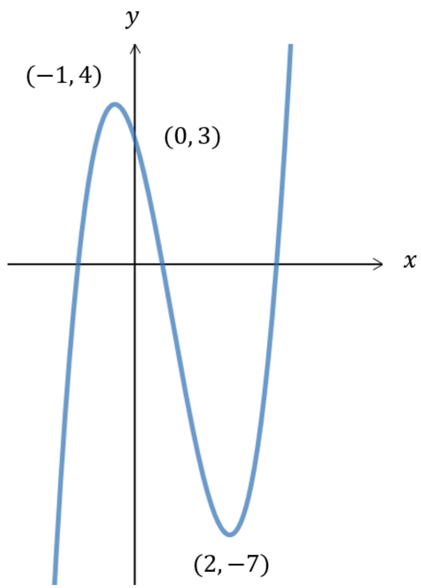
Correct sketch

New turning points: $(-5, 51)$ and $(-2, -26)$

| Worked example | Your turn |
| --- | --- |
| A sketch of the graph $y = f(x)$ is shown:  Sketch the graph of $y = -5f(x+2) + 3$ | A sketch of the graph $y = f(x)$ is shown:  Sketch the graph of $y = -7f(x-3) - 2$<br>Correct sketch<br>New turning points: $(1, -51)$ and $(4, 26)$ |

## Worked example

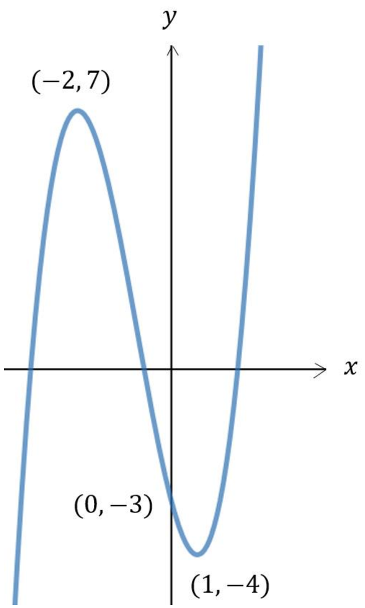
A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = |f(x)|$

## Your turn

A sketch of the graph $y = f(x)$ is shown:

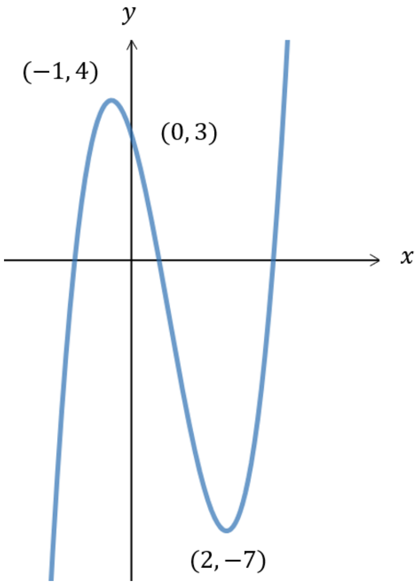



Sketch the graph of $y = |f(x)|$

Correct sketch
New $y$-intercept: $(0, 3)$
New turning points: $(-2, 7)$ and $(1, 4)$

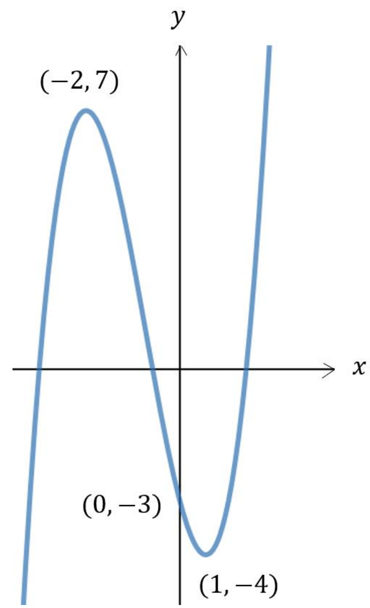
| Worked example | Your turn |
| --- | --- |

A sketch of the graph $y = f(x)$ is shown:




Sketch the graph of $y = |f(-x)|$

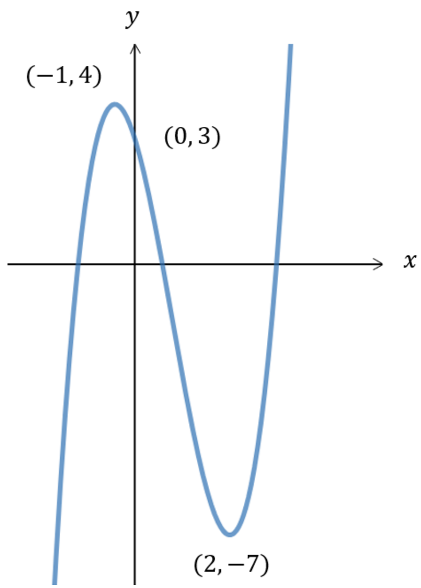
A sketch of the graph $y = f(x)$ is shown:



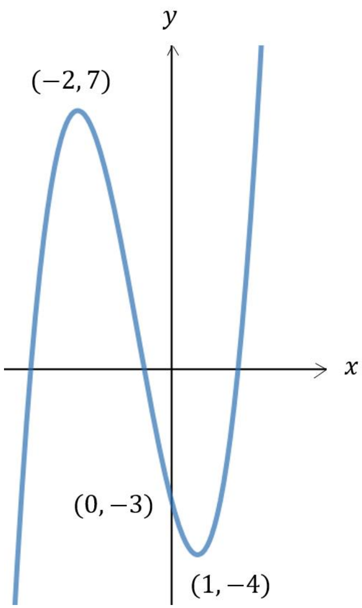

Sketch the graph of $y = |f(-x)|$

Correct sketch
New $y$-intercept: $(0, 3)$
New turning points: $(-1, 4)$ and $(2, 7)$

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |



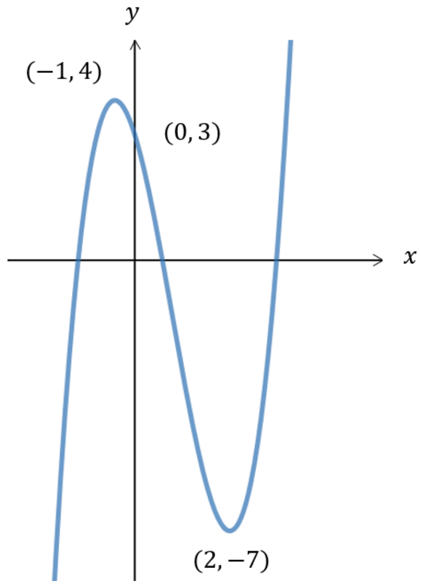

Sketch the graph of $y = f(|x|)$



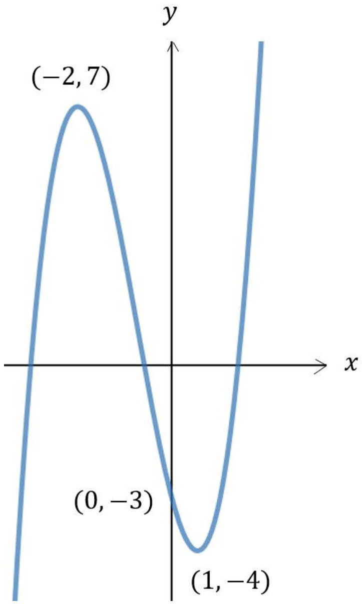

Sketch the graph of $y = f(|x|)$

Correct sketch
New $y$-intercept: $(0, -3)$
New turning points: $(-1, -4)$ and $(1, -4)$

| Worked example | Your turn |
|---|---|
| A sketch of the graph $y = f(x)$ is shown: | A sketch of the graph $y = f(x)$ is shown: |




Sketch the graph of $y = -f(|x|)$

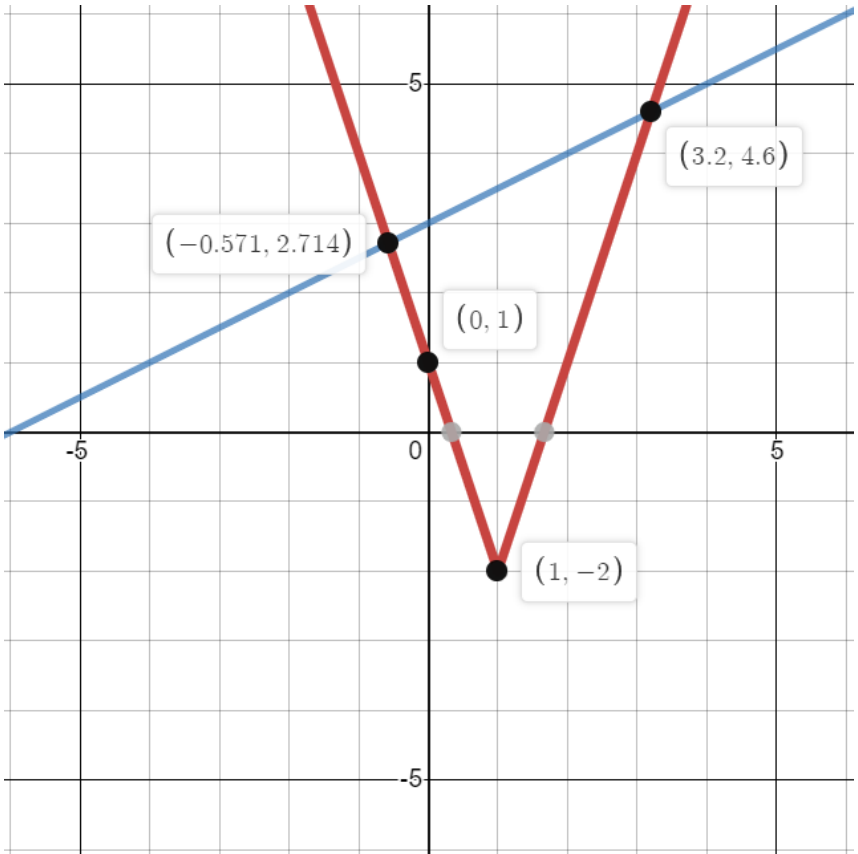



Sketch the graph of $y = -f(|x|)$

Correct sketch
New $y$-intercept: $(0, 3)$
New turning points: $(-1, 4)$ and $(1, 4)$

# 2.7) Solving modulus problems

| Worked example | Your turn |
| --- | --- |
| $f(x) = 2\|x+1\| - 3, x \in \mathbb{R}$<br>(a) Sketch the graph of $y = f(x)$<br>(b) State the range of $f$.<br>(c) Solve the equation $f(x) = \frac{1}{3}x + 2$ | $p(x) = 3\|x-1\| - 2, x \in \mathbb{R}$<br>(a) Sketch the graph of $y = p(x)$<br>(b) State the range of $p$.<br>(c) Solve the equation $p(x) = \frac{1}{2}x + 3$<br>(a) Sketch<br>(b) $p(x) \geq -2$<br>(c) $x = -\frac{4}{7}, x = \frac{16}{5}$ |

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# Worked example

$$f(x) = 6 - 2|x + 3|, x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

(b) State the range of $f$.

(c) Solve the inequality $f(x) > 5$

# Your turn

$$p(x) = 6 - 2|x + 3|, x \in \mathbb{R}$$

(a) Sketch the graph of $y = p(x)$

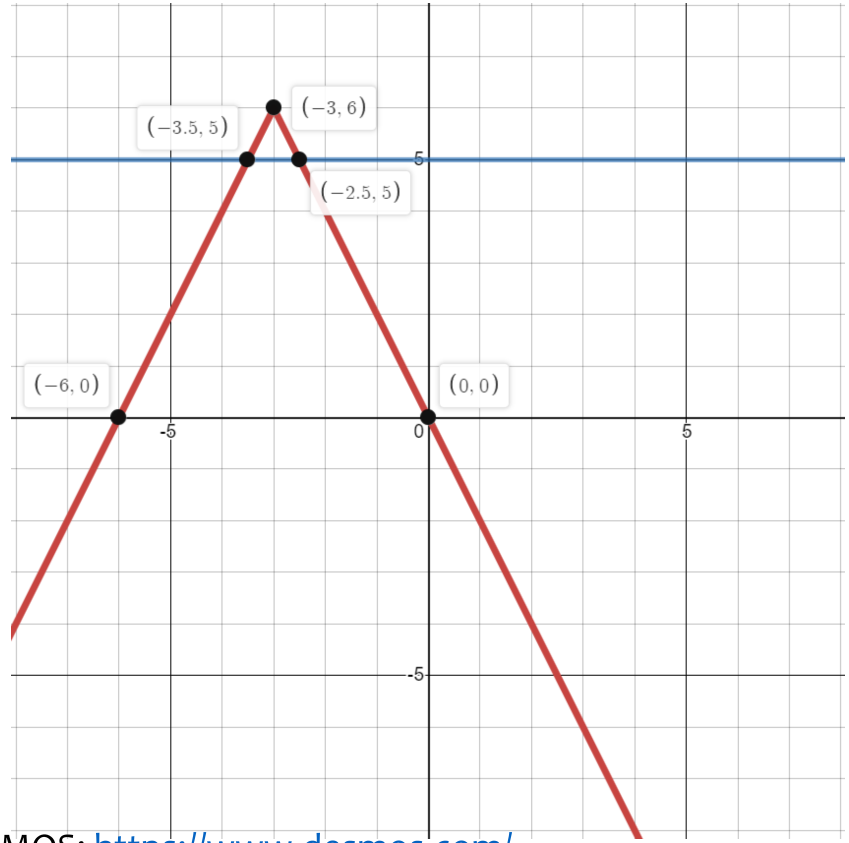
(b) State the range of $p$.

(c) Solve the inequality $p(x) > 5$

(a) Sketch

(b) $p(x) \leq 6$

(c) $-\frac{7}{2} < x < -\frac{5}{2}$

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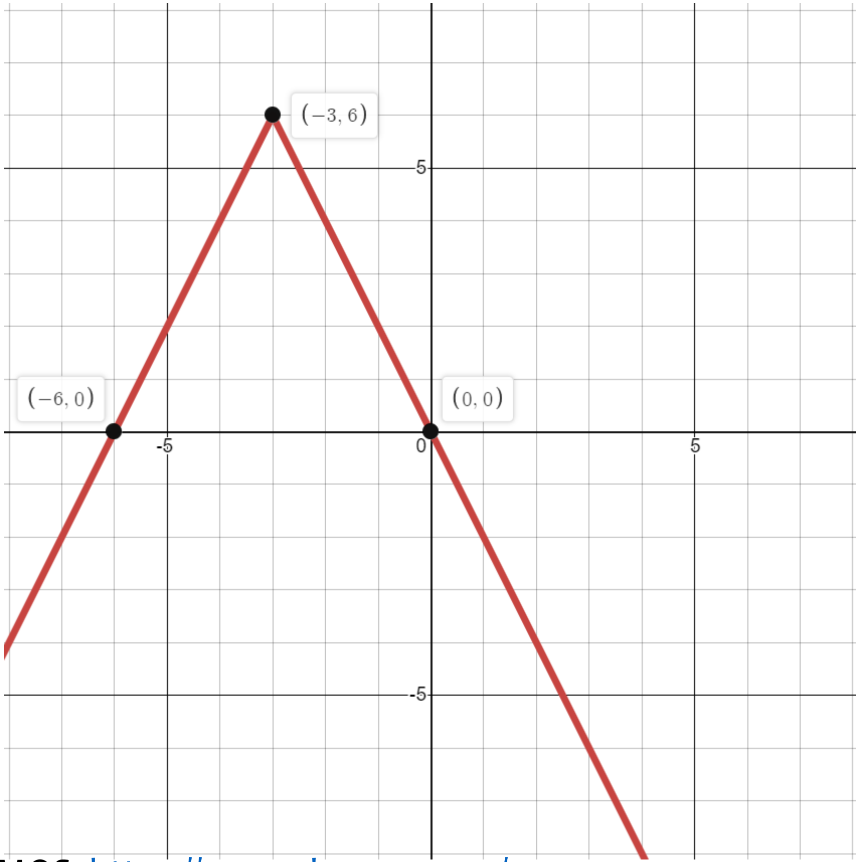
| Worked example | Your turn |
|---|---|
| $f(x) = 6 + 3\|x - 2\|, x \in \mathbb{R}$<br>State the range of values of $k$ for which $f(x) = k$ has:<br>a) no solutions<br>b) exactly one solution<br>c) two distinct solutions | $h(x) = 6 - 2\|x + 3\|, x \in \mathbb{R}$<br>State the range of values of $k$ for which $f(x) = k$ has:<br>a) no solutions<br>b) exactly one solution<br>c) two distinct solutions<br>a) $k > 6$<br>b) $k = 6$<br>c) $k < 6$ |

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