14) Exponentials and logarithms

14.1) Exponential functions
14.2) $y = e^x$
14.3) Exponential modelling
14.4) Logarithms
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14.7) Working with natural logarithms
14.8) Logarithms and non-linear data

14.1) Exponential functions

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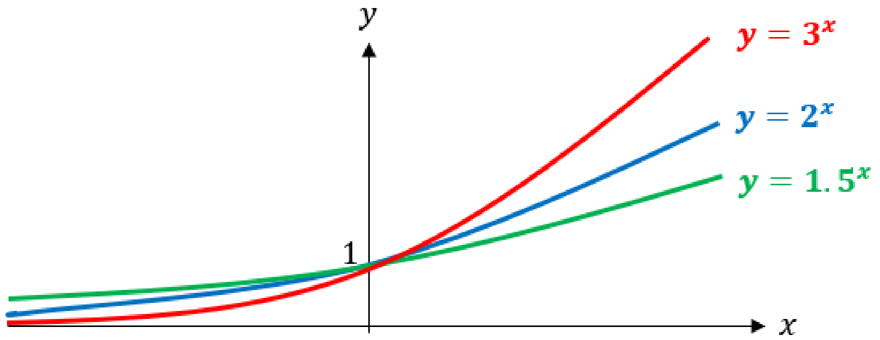
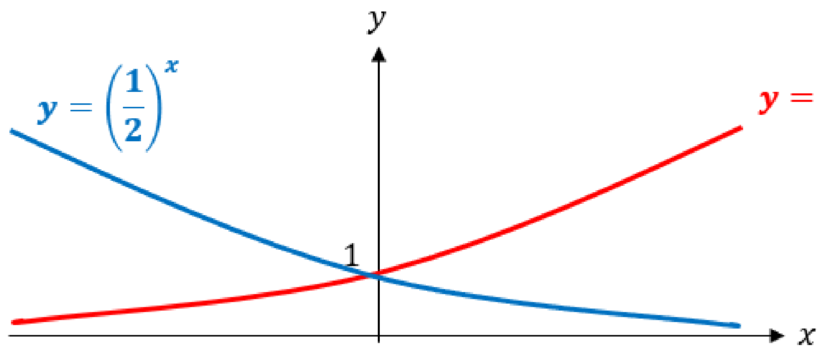
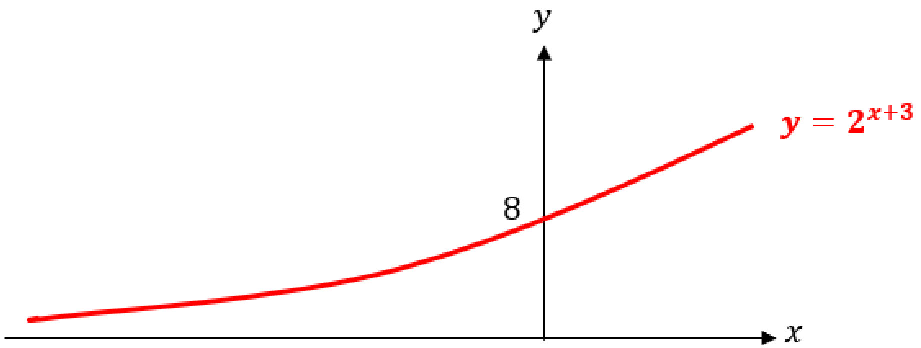
Worked example	Your turn
On the same axes, sketch $y = 4^x$, $y = 5^x$ and $y = 3.5^x$	On the same axes, sketch $y = 2^x$, $y = 3^x$ and $y = 1.5^x$ 

Worked example	Your turn
On the same axes, sketch $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$	On the same axes, sketch $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$



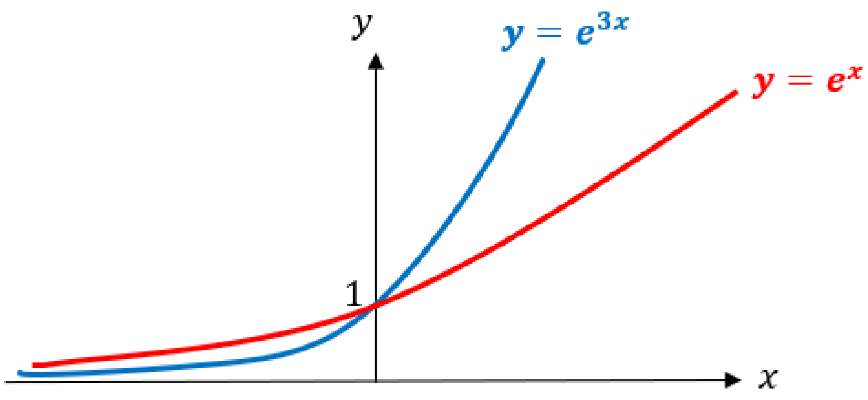
Worked example	Your turn
Sketch $y = 3^{x-2}$	Sketch $y = 2^{x+3}$



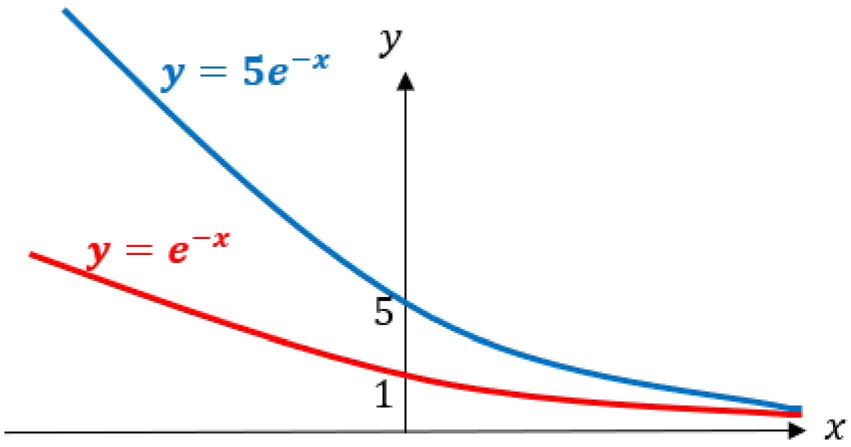
Worked example	Your turn
The graph of $y = ka^x$ passes through the points $(4, \frac{16}{3})$ and $(0, \frac{1}{3})$ Find the values of the constants k and a	The graph of $y = pq^x$ passes through the points $(2, 4.5)$ and $\left(5, \frac{243}{2}\right)$ Find the values of the constants p and q $p = \frac{1}{2}, q = 3$

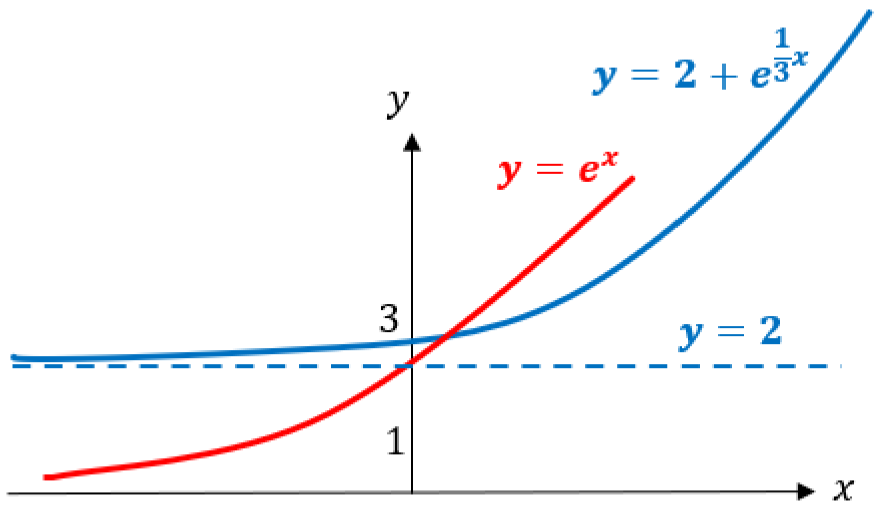
14.2) $y = e^x$

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
e^{2x}	$8e^{-\frac{1}{4}x}$
	$-2e^{-\frac{1}{4}x}$
e^{-3x}	
$4e^{5x}$	
$6e^{\frac{1}{3}x}$	

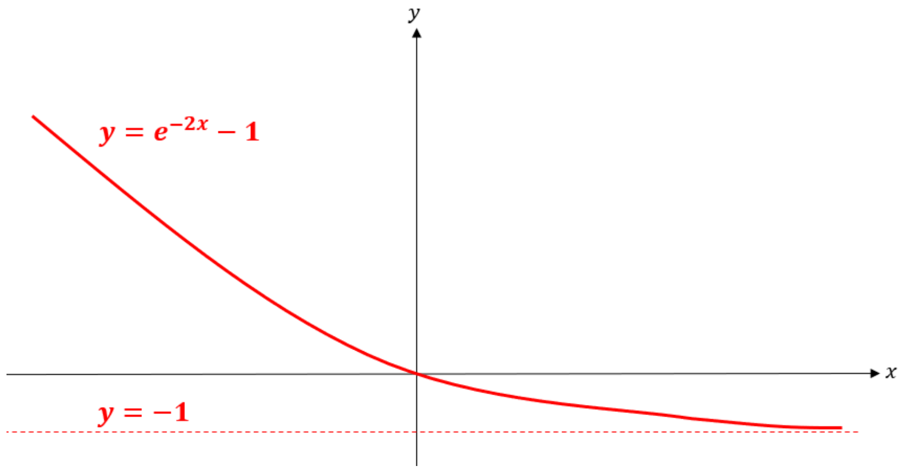
Worked example	Your turn
Sketch: $y = e^{2x}$	Sketch: $y = e^{3x}$ 

Worked example	Your turn
Sketch: $y = 10e^{-x}$	Sketch: $y = 5e^{-x}$



Worked example	Your turn
Sketch: $$y = 3 + 4e^{\frac{1}{2}x}$$	Sketch: $$y = 2 + e^{\frac{1}{3}x}$$ 

Worked example	Your turn
Sketch: $y = e^{-3x} - 2$	Sketch: $y = e^{-2x} - 1$



14.3) Exponential modelling

Worked example	Your turn
Suppose the population P of a village is modelled by $P = 500e^{2t}$ where t is the numbers of years since February 2009. Find: a) The initial population b) The initial rate of growth c) The population in February 2014	Suppose the population P of a village is modelled by $P = 100e^{3t}$ where t is the numbers of years since January 2010. Find: a) The initial population b) The initial rate of growth c) The population in January 2014 a) 100 b) $\frac{dP}{dt} = 300$ c) 16275479

Worked example	Your turn
The density of a pesticide in a given section of field, P mg/m², can be modelled by the equation $P = 80e^{-0.003t}$ where t is the time in days since the pesticide was first applied. a. Use this model to estimate the density of pesticide after 30 days. b. Interpret the meaning of the value 80 in this model. c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k. d. Interpret the significance of the sign of your answer in part (c). e. Sketch the graph of P against t.	The density of a pesticide in a given section of field, P mg/m², can be modelled by the equation $P = 160e^{-0.006t}$ where t is the time in days since the pesticide was first applied. a. Use this model to estimate the density of pesticide after 15 days. b. Interpret the meaning of the value 160 in this model. c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k. d. Interpret the significance of the sign of your answer in part (c). e. Sketch the graph of P against t.

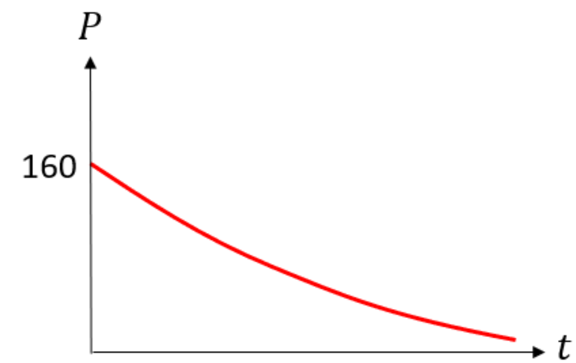
a) $145.2\ mg/m^2$

b) 160 is the initial density of pesticide in the field.

c) $\frac{dP}{dt} = -0.96e^{-0.006t}$; $k = -0.96$

d) The rate is negative, thus the density of pesticide is decreasing.

e)



14.4) Logarithms

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Worked example	Your turn
Write each statement as a logarithm:	Write each statement as a logarithm:
$2^3 = 8$	$3^2 = 9$
	$\log_3 9 = 2$
$7^2 = 49$	$2^7 = 128$
	$\log_2 128 = 7$
$64^{\frac{1}{3}} = 4$	$64^{\frac{1}{2}} = 8$
	$\log_{64} 8 = \frac{1}{2}$

Worked example	Your turn
Write each statement using a power:	Write each statement using a power:
$\log_4 64 = 3$	$\log_3 81 = 4$
	$3^4 = 81$
$\log_3 \frac{1}{9} = -2$	$\log_2 \frac{1}{8} = -3$
	$2^{-3} = \frac{1}{8}$

Worked example	Your turn
Without a calculator, find the value of:	Without a calculator, find the value of:
$\log_5 125$	$\log_4 16$ 2
$\log_5 5$	$\log_4 1$ 0
$\log_5\left(\frac{1}{625}\right)$	$\log_4 4$ 1
$\log_5 1$	$\log_4\left(\frac{1}{64}\right)$ -3

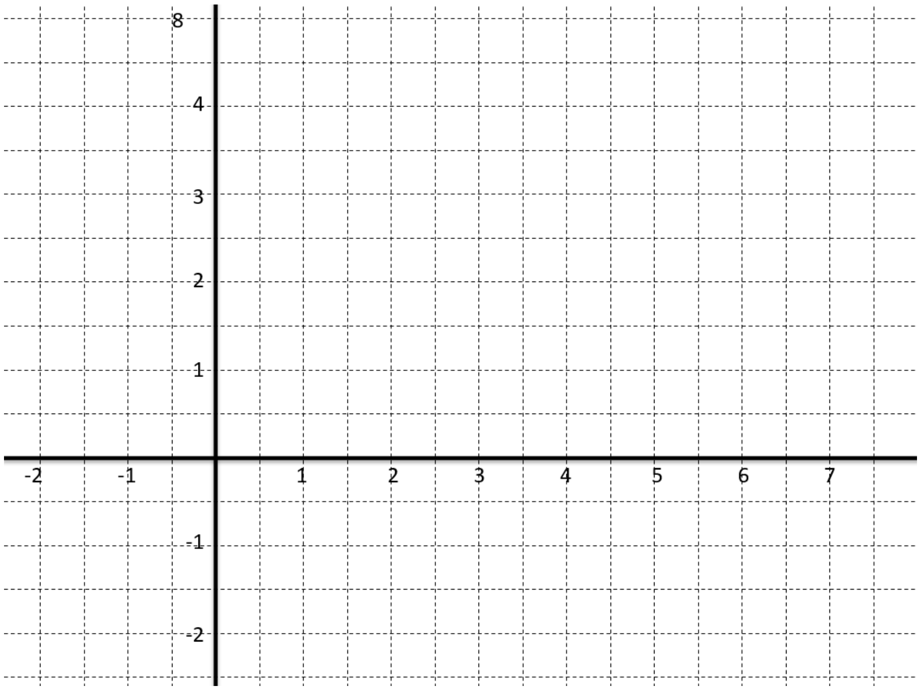
Worked example	Your turn
Without a calculator, find the value of:	Without a calculator, find the value of:
$\log_5 125$	$\log_4 16$ 2
$\log_5 5$	$\log_4 1$ 0
$\log_5\left(\frac{1}{625}\right)$	$\log_4 4$ 1
$\log_5 1$	$\log_4\left(\frac{1}{64}\right)$ -3
$\log_5(-2)$	$\log_4(-3)$ No value

Worked example	Your turn
Without a calculator, find the value of:	Without a calculator, find the value of:
$\log_5 5$	$\log_3 3$
	1
$\ln e^2$	$\ln e$
	1
$\log 1000$	$\log 100$
	2

Worked example	Your turn
Use your calculator to find the value of:	Use your calculator to find the value of:
$\log_5 40$	$\log_3 40$
	3.358 (3 dp)
$\ln 16$	$\ln 8$
	2.079 (3 dp)
$\log 25$	$\log 75$
	1.875 (3 dp)

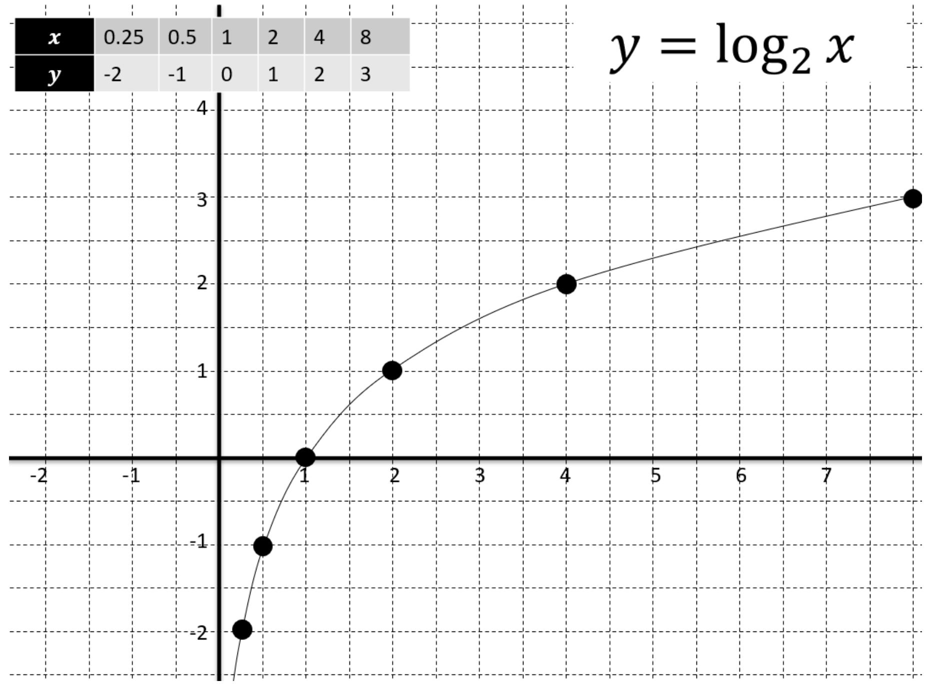
Worked example	Your turn

Using a table of values sketch the graph of $y = \log_4 x$



Using a table of values sketch the graph of $y = \log_2 x$

x	0.25	0.5	1	2	4	8
y	-2	-1	0	1	2	3



14.5) Laws of logarithms

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Worked example	Your turn
Write as a single logarithm:	Write as a single logarithm:
a) $\log_2 9 + \log_2 6$	a) $\log_3 6 + \log_3 7$
b) $\log_3 48 - \log_3 6$	b) $\log_2 15 - \log_2 3$
c) $3\log_4 2 + 2\log_4 5$	c) $2\log_5 3 + 3\log_5 2$
d) $\log_6 7 - 3\log_{10}\left(\frac{1}{4}\right)$	d) $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$
	a) $\log_3 42$
	b) $\log_2 5$
	c) $\log^5 72$
	d) $\log_{10} 48$

Worked example	Your turn
Write as a single logarithm: a) $\log_a(x^4y^5z)$ b) $\log_a\left(\frac{x^3}{y^2}\right)$ c) $\log_a\left(\frac{x^3\sqrt[3]{y}}{z^2}\right)$ d) $\log_a\left(\frac{x^2}{a^5}\right)$	Write as a single logarithm: a) $\log_a(x^2yz^3)$ b) $\log_a\left(\frac{x}{y^3}\right)$ c) $\log_a\left(\frac{x\sqrt{y}}{z}\right)$ d) $\log_a\left(\frac{x}{a^4}\right)$ a) $2\log_a x + \log_a y + 3\log_a z$ b) $\log_a x - 3\log_a y$ c) $\log_a x + \frac{1}{2}\log_a(y) - \log_a z$ d) $\log_a x - 4$

Worked example	Your turn
Solve the equation: $$\log_{10} 2 + 4\ \log_{10} x = 2.209515015$$	Solve the equation: $$\log_{10} 4 + 2 \log_{10} x = 2$$ $$x = 5$$

Worked example	Your turn
Solve the equation: $\log_2(x+5) - \log_2(x-11) = 3$	Solve the equation: $\log_3(x+11) - \log_3(x-5) = 2$ $x = 7$

Worked example	Your turn
Solve the equation: $$2\log_4(x+3) - \log_4 x = 2$$	Solve the equation: $$2\log_2(x+15) - \log_2 x = 6$$ $$x = 25, x = 9$$

14.6) Solving equations using logarithms

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Worked example	Your turn
Solve the equation: $2^x = 30$ $5^x = 0.1$	Solve the equation: $3^x = 20$ $x = 2.727$ (3 dp)

Worked example	Your turn
Solve the equation: $$4^{5x-1} = 71$$ $$3^{1-4x} = 17$$	Solve the equation: $$5^{4x-1} = 61$$ $x = 0.889$ (3 dp)

Worked example	Your turn
Solve the equation: $$2^x = 3^{x+1}$$	Solve the equation: $$3^x = 2^{x+1}$$ $x = 1.7095$ (4 dp)

Worked example	Your turn
Solve the equation: $3^{2x} - 9(3^x) + 14 = 0$	Solve the equation: $5^{2x} - 15(5^x) + 20 = 0$ $x = 1.43, x = 0.431$ (3 sf)

Worked example	Your turn
Solve the equation: $$3^x 2^{x+1} = 7$$ Give your answer in exact form	Solve the equation: $$2^x 3^{x+1} = 5$$ Give your answer in exact form $$x = \frac{\log 5 - \log 3}{\log 6}$$

Worked example	Your turn
Solve the equation: $$2^{x-1} = 5^{x+1}$$ Round your answer to 3 decimal places	Solve the equation: $$3^{x+1} = 4^{x-1}$$ Round your answer to 3 decimal places $x = 8.638$ (3 dp)

14.7) Working with natural logarithms

Worked example	Your turn
Solve the equation: $e^x = 2$ $e^x = 4$	Solve the equation: $e^x = 5$ $x = \ln 5 = 1.609$ (3 dp)

Worked example	Your turn
Solve the equation: $$e^{7x-2} = 3$$ Give your answer in exact form	Solve the equation: $$e^{2x+3} = 7$$ $$x = \frac{1}{2}\ln 7 - \frac{3}{2}$$

Worked example	Your turn
Solve the equation: $$e^{2x} + 2e^{x} = 15$$	Solve the equation: $$e^{2x} + 5e^{x} = 14$$ $$x = \ln 2$$

Worked example	Your turn
Solve the equation: $$e^x - 12e^{-x} = -1$$	Solve the equation: $$e^x - 2e^{-x} = 1$$ $$x = \ln 2$$

Worked example	Your turn
Solve the equation: $$3^x e^{x+4} = 2$$ Give your answer as an exact value	Solve the equation: $$2^x e^{x+1} = 3$$ Give your answer as an exact value $$x = \frac{\ln 3 - 1}{\ln 2 + 1}$$

Worked example

Solve the equation:

$$\ln x = 2$$

$$\ln x = 4$$

Your turn

Solve the equation:

$$\ln x = 5$$

$$x = e^5 = 148.413 \text{ (3 dp)}$$

Worked example	Your turn
Solve the equation: $$3\ln x - 7 = 5$$	Solve the equation: $$2\ln x + 1 = 5$$ $x = e^2 = 7.389$ (3 dp)

Worked example	Your turn
Solve the equation: $$\ln(2x-3)=1$$	Solve the equation: $$\ln(3x+1)=2$$ $$x=\frac{e^2-1}{3}$$

Worked example	Your turn
Find the exact coordinates of the points where the graph with equation $y = 6 + \ln(5 - x)$ intersects the axes	Find the exact coordinates of the points where the graph with equation $y = 2 + \ln(3 - x)$ intersects the axes $(5 - e^{-6}, 0)$ and $(0, 2 + \ln 3)$

14.8) Logarithms and non-linear data

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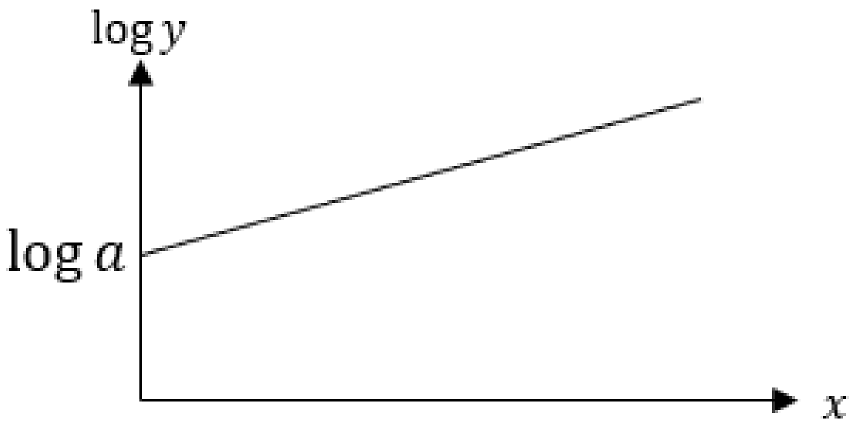
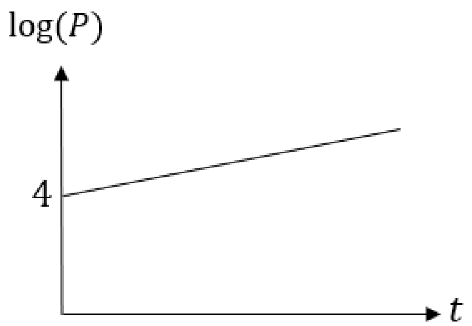
Worked example	Your turn
Use logarithms to convert the non-linear relationship into a linear form and sketch the resulting straight line. $$y = ax^n$$	Use logarithms to convert the non-linear relationship into a linear form and sketch the resulting straight line. $$y = ab^x$$ $$\log y = (\log b)\,x + \log a$$ 

Worked example

The graph represents the growth of a population of bacteria, P, over t hours.
The graph has a gradient of 0.3 and meets the vertical axis at $(0,4)$.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

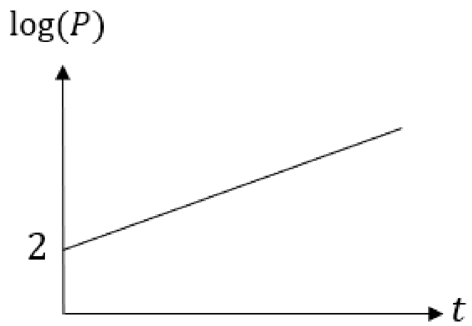
a. Write down an equation for the line.
b. Find the values of a and b, giving them to 3 sf where necessary.
c. Interpret the meaning of the constant a in this model.



Your turn

The graph represents the growth of a population of bacteria, P, over t hours.
The graph has a gradient of 0.6 and meets the vertical axis at $(0,2)$.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a. Write down an equation for the line.
b. Find the values of a and b, giving them to 3 sf where necessary.
c. Interpret the meaning of the constant a in this model.



a) $\log P = 0.6t + 2$
b) $a = 100$, $b = 3.98$ (3 sf)
c) The initial size of the bacteria population was 100

Worked example

The table below gives the rank (by size) and population of a country's largest cities and districts (the capital city is number 1 but has been excluded as an outlier).

City	**A**	**B**	**C**	**D**	**E**
Rank, R	2	3	4	5	6
Population	2 000 000	1 400 000	1 200 000	1 000 000	900 000

The relationship between the rank and population can be modelled by the formula:

$P = aR^n$ where a and n are constants.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.

b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.

c) Use your graph to estimate the values of a and n to two significant figures.

Your turn

The table below gives the rank (by size) and population of the UK's largest cities and districts in the past (London is number 1 but has been excluded as an outlier).

City	**Birmingham**	**Leeds**	**Glasgow**	**Sheffield**	**Bradford**
Rank, R	2	3	4	5	6
Population	1 000 000	730 000	620 000	530 000	480 000

The relationship between the rank and population can be modelled by the formula:

$P = aR^n$ where a and n are constants.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.

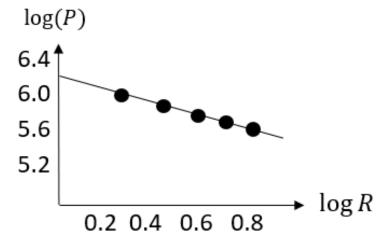
b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.

c) Use your graph to estimate the values of a and n to two significant figures.

a)

$\log R$	0.30	0.48	0.60	0.70	0.78
$\log P$	6	5.86	5.79	5.72	5.68

b)



c) $a = 1600000, n = -0.67$ (2 sf)

Worked example

A population is increasing exponentially according to the model $P = ab^t$, where a, b are constants to be found.

The population is recorded as follows:

Years t after 2016	1.4	2.6	4.4
Population P	4706	7346	14324

a) Draw a table giving values of t and $\log P$ (to 3dp).
b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 1.4$) and last (where $t = 4.4$). Determine the equation of this line of best fit.
c) Hence, determine the values of a and b in the model.
d) Estimate the population in 2020

Your turn

A population is increasing exponentially according to the model $P = ab^t$, where a, b are constants to be found.

The population is recorded as follows:

Years t after 2015	0.7	1.3	2.2
Population P	2353	3673	7162

a) Draw a table giving values of t and $\log P$ (to 3dp).
b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 0.7$) and last (where $t = 2.2$). Determine the equation of this line of best fit
c) Hence, determine the values of a and b in the model.
d) Estimate the population in 2020

a)

t	0.7	1.3	2.2
$\log P$	3.372	3.565	3.855

b) $\log P = 0.322t + 3.147$
c) $a = 1403, b = 2.099$ (4 sf)
d) 57164