12) Vectors

12.1) 3D coordinates

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Worked example	Your turn
Find the distance from the origin to the point with coordinates $(6, 8, 24)$	Find the distance from the origin to the point with coordinates $(-3, -4, -12)$ 13
Find the distance from the origin to the point with coordinates $(-6, 0, -2)$	

Worked example

Find the distance between the points:

$A(1, 3, 5)$ and $B(-6, 0, -4)$

$C(-1, 0, 1)$ and $D(0, 0, -3)$

Your turn

Find the distance between the points:

$E(1, 3, 4)$ and $B(8, 6, -5)$

11.8 (1 dp)

Worked example

The coordinates of A and B are $(3,5,-2)$ and $(3,k,-1)$ respectively. Given that the distance from A to B is $\sqrt{2}$ units, find the possible values of k.

Your turn

The coordinates of A and B are $(5,3,-8)$ and $(1,k,-3)$ respectively. Given that the distance from A to B is $3\sqrt{10}$ units, find the possible values of k.

$k = -4$ or $k = 10$

12.2) Vectors in 3D

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Worked example

Consider the points $A(-1, -5, 2)$ and $B(-7, 3, 0)$.

a) Find the position vectors of A and B in ijk notation

b) Find the vector $\overrightarrow{AB}$ as a column vector.

Your turn

Consider the points $A(1, 5, -2)$ and $B(0, -3, 7)$.

a) Find the position vectors of A and B in ijk notation

b) Find the vector $\overrightarrow{AB}$ as a column vector.

a) $\overrightarrow{OA} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{OB} = -\mathbf{3}\boldsymbol{j} + 7\boldsymbol{k}$

b) $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -8 \\ 9 \end{pmatrix}$

Worked example

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by:

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

a) Find:

i) $\boldsymbol{a} + 3\boldsymbol{b}$

ii) $4\boldsymbol{a} - 5\boldsymbol{b}$

b) State whether these vectors are parallel to $-4\boldsymbol{i} + 16\boldsymbol{j}$

Your turn

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by:

$$\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

a) Find:

i) $4\boldsymbol{a} + \boldsymbol{b}$

ii) $2\boldsymbol{a} - 3\boldsymbol{b}$

b) State whether these vectors are parallel to $4\boldsymbol{i} - 5\boldsymbol{k}$

a)

i) $\begin{pmatrix} 12 \\ -14 \\ 20 \end{pmatrix}$

ii) $\begin{pmatrix} -8 \\ 0 \\ 10 \end{pmatrix}$

b)

i) $\begin{pmatrix} 12 \\ -14 \\ 20 \end{pmatrix} = 3\begin{pmatrix} 4 \\ -\frac{14}{3} \\ \frac{20}{3} \end{pmatrix} \neq k\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$ ∴ Not parallel

ii) $\begin{pmatrix} -8 \\ 0 \\ 10 \end{pmatrix} = -2\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} = k\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$ $(k = -2)$ ∴ Parallel

Worked example

Find the magnitude of the vector $\begin{pmatrix} 6 \\ 8 \\ 24 \end{pmatrix}$

Your turn

Find the magnitude of the vector $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

13

Worked example

Find the magnitude of the vector

$$\boldsymbol{a} = 6\boldsymbol{i} - 8\boldsymbol{j} + 24\boldsymbol{k}$$

and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

Find the magnitude of the vector

$$\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$$

and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

Your turn

Find the magnitude of the vector

$$\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$$

and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

$$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{21}}(2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$$

Worked example

Find the angles that the vector

$$\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

makes with each of the positive coordinate axes. Give your answers to 1 decimal place.

Your turn

Find the angles that the vector

$$\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$$

makes with each of the positive coordinate axes. Give your answers to 1 decimal place.

$$\theta_x = 57.7°$$
$$\theta_y = 143.3°$$
$$\theta_z = 105.5°$$

Worked example

The points A and B have position vectors $\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$ and $-2\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}$ relative to a fixed origin, O.
Show that ΔOAB is isosceles.

Your turn

The points A and B have position vectors $4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ relative to a fixed origin, O.
Show that ΔOAB is isosceles.

$$\left|\overrightarrow{AB}\right| = \sqrt{69}$$
$$\left|\overrightarrow{OA}\right| = \sqrt{69}$$
$$\left|\overrightarrow{OB}\right| = \sqrt{26}$$
$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{OA}\right| \neq \left|\overrightarrow{OB}\right|$$
$$\therefore\ OAB \text{ is isosceles.}$$

Worked example

$\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$

By considering the angles that $\boldsymbol{a}$ and $\boldsymbol{b}$ make with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

Your turn

$\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

By considering the angles that $\boldsymbol{a}$ and $\boldsymbol{b}$ make with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

2.90 (3 sf)

Worked example

A triangle PQR is such that
$\overrightarrow{PQ} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{QR} = 4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$
Find $< PQR$ to 1 decimal place

Your turn

A triangle PQR is such that
$\overrightarrow{PQ} = 2\boldsymbol{i} - \mathbf{3}\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{QR} = -4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$
Find $< PQR$ to 1 decimal place

$41.9°$

12.3) Solving geometric problems

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Worked example

A, B, C and D are the points $(3, -4, -9)$, $(1, -7, -3)$, $(1, 0, -15)$ and $(7, 9, -33)$ respectively.

a) Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.

b) Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 3\overrightarrow{AB}$.

c) Hence describe the quadrilateral $ABCD$.

Your turn

A, B, C and D are the points $(2, -5, -8)$, $(1, -7, -3)$, $(0, 15, -10)$ and $(2, 19, -20)$ respectively.

a) Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.

b) Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 2\overrightarrow{AB}$.

c) Hence describe the quadrilateral $ABCD$.

a)

$\overrightarrow{AB} = -\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$

$\overrightarrow{DC} = -2\boldsymbol{i} - 4\boldsymbol{j} + 10\boldsymbol{k}$

b)

$\overrightarrow{DC} = 2(-\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) = 2\overrightarrow{AB}$

They are multiples ∴ parallel.

c)

AB and DC are parallel but different in length. Therefore $ABCD$ is a trapezium.

Worked example

P, Q and R are the points $(9,3,-4)$, $(-5,5,5)$ and $(0,2,-8)$ respectively.

Find the coordinates of the point S so that $PQRS$ forms a parallelogram.

Your turn

P, Q and R are the points $(4,-9,-3)$, $(7,-7,-7)$ and $(8,-2,0)$ respectively.

Find the coordinates of the point S so that $PQRS$ forms a parallelogram.

$S(5,-4,4)$

Worked example

Given that
$(q-5)\boldsymbol{i}+2\boldsymbol{j}-120\boldsymbol{k}=p\boldsymbol{i}+q\boldsymbol{j}+4pqr\boldsymbol{k}$,
find the values of p, q and r.

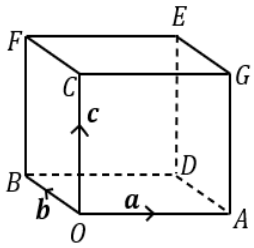
Your turn

Given that
$3\boldsymbol{i}+(p+2)\boldsymbol{j}+120\boldsymbol{k}=p\boldsymbol{i}-q\boldsymbol{j}+4pqr\boldsymbol{k}$,
find the values of p, q and r.

$$p=3, q=-5, r=-2$$

Worked example

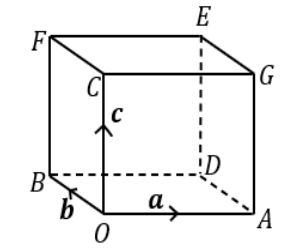
The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G.
Vectors a, b and c are the position vectors of the vertices A, B and C respectively.
Prove that the diagonals OE and AF bisect each other.



Your turn

The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G.
Vectors a, b and c are the position vectors of the vertices A, B and C respectively.
Prove that the diagonals OE and BG bisect each other.

Proof



Diagrams/Graphs used with permission from DrFrostMaths: https://www.drfrostmaths.com/

12.4) Application to mechanics

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Worked example

Convert these vectors to scalar form:

- A force of $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} N$
- An acceleration of $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} ms^{-2}$
- A displacement of $\begin{pmatrix} -6 \\ 8 \\ -24 \end{pmatrix} m$
- A velocity of $\begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix} ms^{-1}$

Your turn

Convert these vectors to scalar form:

- A force of $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} N$

  A force of 5.10 N (3 sf)

- An acceleration of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ms^{-2}$

  An acceleration of 1.41 ms^{-2} (3 sf)

- A displacement of $\begin{pmatrix} 12 \\ -3 \\ 4 \end{pmatrix} m$

  A distance of 13 m

- A velocity of $\begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} ms^{-1}$

  A speed of 5 ms^{-1}

Worked example

A particle of mass 0.25 kg is acted on by three forces.

$$F_1 = (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})\ N$$
$$F_2 = (2\boldsymbol{i} - 4\boldsymbol{k})\ N$$
$$F_3 = (-5\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k})\ N$$

a) Find the resultant force R acting on the particle.

b) Find the acceleration of the particle, giving your answer in the form $(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k})$ ms^{-2}.

c) Find the magnitude of the acceleration.

Given that the particle starts at rest,

d) Find the distance travelled by the particle in the first 3 seconds of its motion.

Your turn

A particle of mass 0.5 kg is acted on by three forces.

$$F_1 = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\ N$$
$$F_2 = (-\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})\ N$$
$$F_3 = (4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})\ N$$

a) Find the resultant force R acting on the particle.

b) Find the acceleration of the particle, giving your answer in the form $(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k})$ ms^{-2}.

c) Find the magnitude of the acceleration.

Given that the particle starts at rest,

d) Find the distance travelled by the particle in the first 6 seconds of its motion.

a) $\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} N$

b) $\boldsymbol{a} = (10\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k})ms^{-2}$

c) $|\boldsymbol{a}| = \sqrt{140}\ ms^{-2} = 11.83\ ms^{-2}$ (2 dp)

d) $36\sqrt{35}\ m = 212.98\ m$ (2 dp)