12) Differentiation

12.1) Gradients of curves

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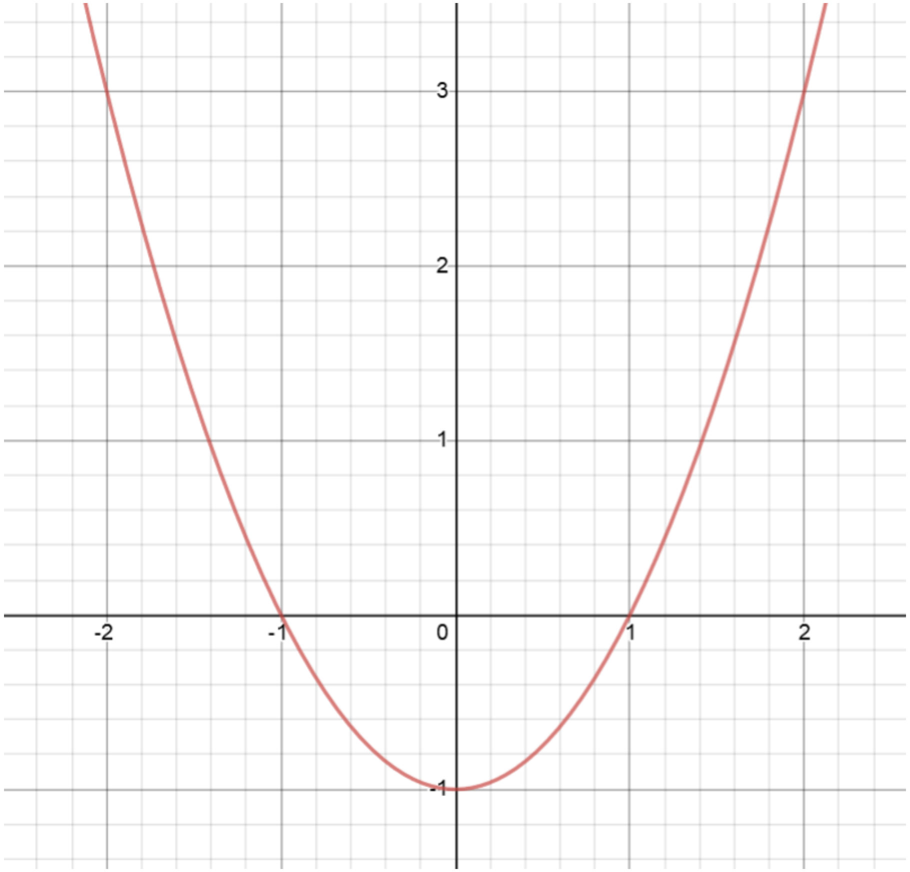
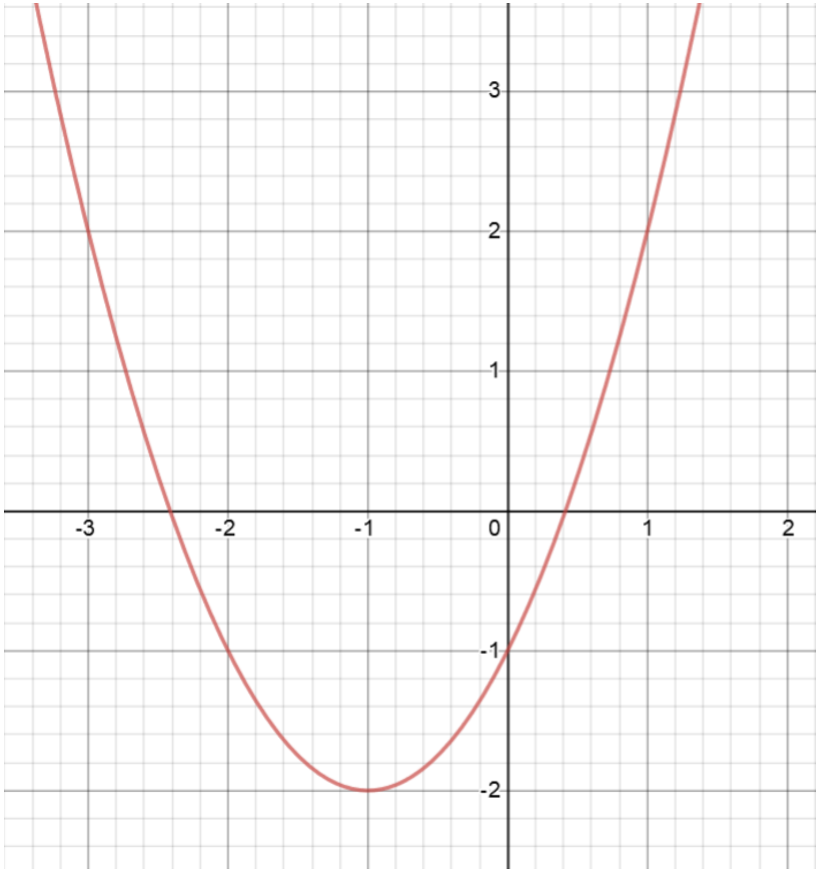
Worked example	Your turn
Find the gradient between the points on the curve $y = x^2$: $(5, 25)$ and $(6, 36)$	Find the gradient between the points on the curve $y = x^2$: $(5, 25)$ and $(5.1, 26.01)$ 10.1

Worked example	Your turn
Find an estimate for the gradient of each curve at the point where: $x = 2$ $x = 0$ $x = -1$	Find an estimate for the gradient of each curve at the point where: $x = 1$ $\quad 4$ $x = -1$ $\quad 0$ $x = -2$ $\quad -0.4$





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12.2) Finding the derivative

Worked example	Your turn
The point A with coordinates $(8,64)$ lies on the curve with equation $y = x^2$. At point A the curve has gradient g. a) Show that $g = \lim_{h \to 0}(16 + h)$ b) Deduce the value of g.	The point A with coordinates $(4,16)$ lies on the curve with equation $y = x^2$. At point A the curve has gradient g. a) Show that $g = \lim_{h \to 0}(8 + h)$ b) Deduce the value of g. Shown

Worked example	Your turn
Prove from first principles that the derivative of $3x$ is 3	Prove from first principles that the derivative of $5x$ is 5

$$f(x) = 5x$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{5(x+h)-5(x)}{h} \\ &= \lim_{h\to 0}\frac{5x+5h-5x}{h} \\ &= \lim_{h\to 0}\frac{5h}{h} \\ &= \lim_{h\to 0} 5 \\ &= 5 \end{aligned}$$

Worked example	Your turn
Prove from first principles that the derivative of $3x^2$ is $6x$	Prove from first principles that the derivative of $5x^2$ is $10x$

$$f(x) = 5x^2$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{5(x+h)^2-5(x)^2}{h} \\ &= \lim_{h\to 0}\frac{5x^2+10xh+5h^2-5x^2}{h} \\ &= \lim_{h\to 0}\frac{10xh+5h^2}{h} \\ &= \lim_{h\to 0}\frac{h(10x+5h)}{h} \\ &= \lim_{h\to 0}(10x+5h) \\ &= 10x \end{aligned}$$

[As $h \to 0, 5h \to 0$]

Worked example	Your turn
Prove from first principles that the derivative of x^4 is $4x^3$.	Prove from first principles that the derivative of x^3 is $3x^2$

$$f(x) = x^3$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - (x)^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh^2 + h^3)}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2$$

[As $h \to 0, 3xh \to 0$ and $h^2 \to 0$]

12.3) Differentiating x^n

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
x^2	x^5
	$5x^4$
x^3	
x^4	

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$3x^2$	$-3x^5$
	$-15x^4$
$-2x^3$	
$5x^4$	

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$\sqrt{x}$	$\sqrt[5]{x}$
	$\frac{1}{5}x^{-\frac{4}{5}}$
$\sqrt[3]{x}$	
$\sqrt[4]{x}$	

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$\frac{1}{x}$	$\frac{1}{x^4}$
	$-4x^{-5} = -\frac{4}{x^5}$
$\frac{1}{x^2}$	
$\frac{1}{x^3}$	

Worked example	Your turn
Differentiate with respect to x: $$\frac{2}{x}$$ $$\frac{3}{4x^2}$$ $$\frac{6}{5x^3}$$	Differentiate with respect to x: $$\frac{7}{8x^4}$$ $$-\frac{7}{2}x^{-5} = -\frac{7}{2x^5}$$

Worked example	Your turn
Differentiate with respect to x: $$\frac{2}{3}\sqrt{x}$$ $$\frac{4}{7}\sqrt[3]{x}$$ $$\frac{5}{6}\sqrt[4]{x}$$	Differentiate with respect to x: $$\frac{3}{5}\sqrt{x}$$ $$\frac{3}{10}x^{-\frac{1}{2}}=\frac{3}{10\sqrt{x}}$$

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$$\frac{2}{3\sqrt{x}}$$	$$\frac{3}{5\sqrt{x}}$$
	$$-\frac{3}{10}x^{-\frac{3}{2}} = -\frac{3}{10x\sqrt{x}}$$
$$\frac{4}{7\sqrt[3]{x}}$$	
$$\frac{5}{6\sqrt[4]{x}}$$	

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$\sqrt{36x^7}$	$\sqrt{16x^8}$
	$16x^3$
$\sqrt{25}x^7$	$\sqrt{9}x^8$
	$24x^7$

12.4) Differentiating quadratics

Worked example	Your turn
Find the gradient of the curve: $y = x^2 + 3x + 2$ at $(4, 30)$ $y = 2x^3 - x + 5$ at $(-1, 4)$	Find the gradient of the curve: $y = 3x^2 - 2x + 1$ at $(-2, 17)$ -14

Worked example	Your turn
Find the coordinates of the point(s) where the gradient is 4: $y = x^2 - 8x + 3$ $y = 5x^2 - x + 7$	Find the coordinates of the point(s) where the gradient is 3: $y = 3x^2 - 9x + 7$ $(2, 1)$

Worked example	Your turn
Let $f(x) = 8x^2 - 4x - 3$ a) Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$ b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 44. c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 12x + 21$.	Let $f(x) = 4x^2 - 8x + 3$ a) Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$ b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8. c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$. a) -4 b) $(2, 3)$ c) At $(1, -1)$ gradient $= 0$ At $(2, 3)$ gradient $= 8$

12.5) Differentiating functions with two or more terms

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$y = 4x^3 + 3x^2 + 2x + 1$	$y = 5x^4 - 2x^7 + 12345 - x^5$ $\frac{dy}{dx} = 20x^3 - 14x^6 - 5x^4$
$f(x) = x^3 - 2x^5 - 3x^{-2} - 2$	

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$y = 2\sqrt{x} + 3x^{\frac{4}{3}} - \frac{1}{x} + \frac{5}{x^2}$	$y = 3\sqrt{x} + 4x^{\frac{5}{3}} - \frac{5}{x} + \frac{1}{\sqrt[3]{x}}$
	$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{20}{3}x^{\frac{2}{3}} + 5x^{-2} - \frac{1}{3}x^{-\frac{4}{3}}$
$f(x) = 4\sqrt[3]{x} + 2x^{\frac{1}{4}} - \frac{5}{x^3} + \frac{3}{\sqrt{x}} + 6x^{-2}$	

Worked example	Your turn
Differentiate with respect to x: $$y = x^4(x-5)$$ $$f(x) = x^3(x+2)$$	Differentiate with respect to x: $$f(x) = x^2(x-3)$$ $$f'(x) = 3x^2 - 6x$$

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:

Worked example

$$y = \frac{(x+3)^2}{x}$$

$$f(x) = \frac{(3x-2)^2}{5x}$$

Your turn

$$f(x) = \frac{(2x+3)^2}{5x}$$

$$f'(x) = \frac{4}{5} - \frac{9}{5}x^{-2}$$

$$= \frac{4}{5} - \frac{9}{5x^2}$$

Worked example	Your turn
Differentiate with respect to x: $$y = \frac{x^3 + 2}{\sqrt{x}}$$ $$f(x) = \frac{x^2 - 5}{\sqrt[3]{x}}$$	Differentiate with respect to x: $$f(x) = \frac{x^2 + 3}{\sqrt{x}}$$ $$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$$

Worked example	Your turn
Differentiate with respect to x: $$y = \frac{(x+4)^3}{5x^2}$$	Differentiate with respect to x: $$y = \frac{(x+2)^3}{3x^2}$$ $$\frac{dy}{dx} = \frac{1}{3} - 4x^{-2} - \frac{16}{3}x^{-3}$$ $$= \frac{1}{3} - \frac{4}{x^2} - \frac{16}{3x^3}$$

Worked example	Your turn
Differentiate with respect to x: $$y = \frac{3-4x}{2x^2\sqrt{x}}$$	Differentiate with respect to x: $$y = \frac{1+2x}{3x\sqrt{x}}$$ $$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{5}{2}} - \frac{1}{3}x^{-\frac{3}{2}}$$

12.6) Gradients, tangents and normal

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Worked example	Your turn
Find the gradient of the curve: $y = 8\sqrt{x} + \frac{48}{x}$ at $(4, 28)$	Find the gradient of the curve: $y = 5\sqrt{x} - \frac{3}{x}$ at $(16, \frac{317}{16})$
	$\frac{163}{256}$
$y = \frac{3}{x^2} - \frac{18}{\sqrt{x}}$ at $(9, -\frac{161}{27})$	

Worked example	Your turn
Find the coordinates of the point(s) where the gradient is 10: $y = x^3 + 6x^2 - 11x + 7$	Find the coordinates of the point(s) where the gradient is 2: $y = x^3 - 3x^2 - 7x + 8$ $(-1, 11)$ and $(3, -13)$

Worked example	Your turn
For the curve $y = f(x)$, $$\frac{dy}{dx} = 723 + kx^5 + 2k,$$ where k is a constant. When $x = -3$, the gradient of the curve is 241. Find k.	For the curve $y = f(x)$, $$\frac{dy}{dx} = \frac{3}{2} - kx^4 + k,$$ where k is a constant. When $x = -2$, the gradient of the curve is -6. Find k. $$k = \frac{1}{2}$$

Worked example	Your turn
Find the equation of the tangent to the curve $y = x^4$ when $x = 2$	Find the equation of the tangent to the curve $y = x^3$ when $x = 2$ $y - 8 = 12(x - 2)$ $y = 12x - 16$

Worked example	Your turn
Find the equation of the normal to the curve $y = x^4$ when $x = 2$	Find the equation of the normal to the curve $y = x^3$ when $x = 2$ $y - 8 = -\frac{1}{12}(x - 2)$ $y = -\frac{1}{12}x + \frac{49}{6}$

Worked example	Your turn
Find the equation of the tangent to the curve with equation $y = x^3 - 5x^2 - 3x + 2$ at the point $(5, -13)$	Find the equation of the tangent to the curve with equation $y = x^3 - 3x^2 + 2x - 1$ at the point $(3, 5)$ $y = 11x - 28$

Worked example	Your turn
Find the equation of the normal to the curve with equation $y = 3 - 4\sqrt[3]{x}$ at the point where $x = 8$. Give your answer in the form $ax + by + c = 0$	Find the equation of the normal to the curve with equation $y = 8 - 3\sqrt{x}$ at the point where $x = 4$. Give your answer in the form $ax + by + c = 0$ $3y - 4x + 10 = 0$

Worked example	Your turn
The point P with x-coordinate $\frac{1}{4}$ lies on the curve with equation $y = 2x^2$. The normal to the curve at P intersects the curve at points P and Q. Find the coordinates of Q	The point P with x-coordinate $\frac{1}{2}$ lies on the curve with equation $y = 4x^2$. The normal to the curve at P intersects the curve at points P and Q. Find the coordinates of Q $$\left(-\frac{9}{16}, \frac{81}{64}\right)$$

12.7) Increasing and decreasing functions

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Worked example	Your turn
Show that the function $f(x) = x^3 - 3x^2 + 8x - 5$ is increasing for all real values of x.	Show that the function $f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x. Shown

Worked example	Your turn
Find the interval(s) on which the function $f(x) = x^3 - 6x^2 - 135x + 1$ is increasing.	Find the interval(s) on which the function $f(x) = x^3 + 6x^2 - 135x - 2$ is increasing. $x \leq -9$ and $x \geq 5$

Worked example	Your turn
Show that the function $5 - x(4x^2 + 3)$ is decreasing for all $x \in \mathbb{R}$	Show that the function $3 + 4x(-x^2 - 5)$ is decreasing for all $x \in \mathbb{R}$ Shown

Worked example	Your turn
Find the interval on which the function $f(x) = x^3 - 3x^2 - 9x - 10$ is decreasing.	Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x + 5$ is decreasing. $[-3, 1]$

12.8) Second order derivatives

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Worked example	Your turn
If $y = 5x^3 - \frac{4}{x^3}$, find $\frac{d^2y}{dx^2}$	If $y = 3x^5 + \frac{4}{x^2}$, find $\frac{d^2y}{dx^2}$ $$\frac{d^2y}{dx^2} = 60x^3 + \frac{24}{x^4}$$

Worked example	Your turn
If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$.	If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$. $f''(x) = -\frac{3}{4}x^{-\frac{3}{2}} + \frac{3}{8}x^{-\frac{5}{2}}$

12.9) Stationary points

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Worked example	Your turn
Find the least value of $$f(x) = x^2 + 6x - 9$$	Find the least value of $$f(x) = x^2 - 4x + 9$$ 5

Worked example	Your turn
Find the turning point of $y = \sqrt[4]{x} - 2x$	Find the turning point of $y = \sqrt{x} - x$ $\left(\frac{1}{4}, \frac{1}{4}\right)$

Worked example	Your turn
Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $y = x^2 + 6x - 2$ $y = 2x^3 + 6x^2 - 4$	Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $y = x^3 + 3x^2 - 4$ $(-2, 0)$ and $(0, -4)$

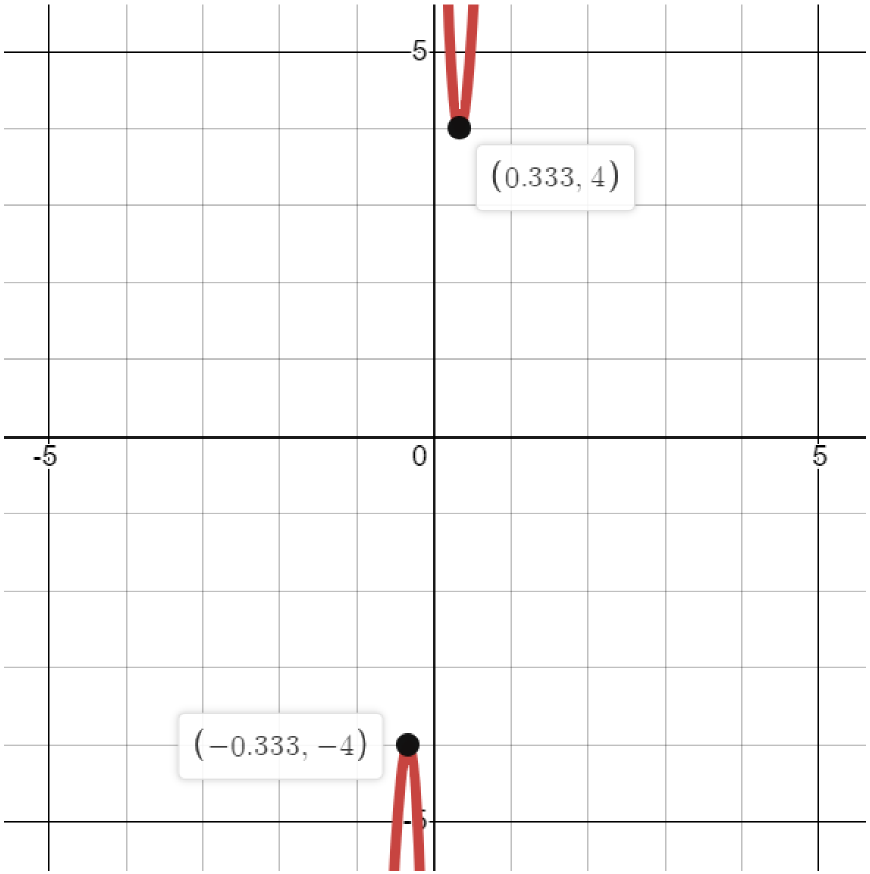
Worked example	Your turn
Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $$y = \frac{2}{3}x^3 - 3.5x^2 + 3x + 5$$	Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $$y = x^3 + \frac{1}{2}x^2 - 2x + 4$$ $\left(-1, \frac{11}{2}\right)$ and $\left(\frac{2}{3}, \frac{86}{27}\right)$

Worked example	Your turn
Find the stationary points on the curve $y = \frac{5}{3}x^3 - 80x$	Find the stationary points on the curve $y = x^3 - 12x$ $(-2, 16)$ and $(2, -16)$

Worked example	Your turn
Find the stationary point on the curve with equation $y = x^4 - 108x$, and determine whether it is a local maximum, a local minimum or a point of inflection.	Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection. $(2, -48)$ Local minimum

Worked example	Your turn
Find the coordinates of the stationary points on the curve with equation $y = 4x^3 + 30x^2 + 48x - 3$ and use the second derivative to determine their nature	Find the coordinates of the stationary points on the curve with equation $y = 2x^3 - 15x^2 + 24x + 6$ and use the second derivative to determine their nature $(1, 17)$ Local maximum $(4, -10)$ Local minimum

Worked example	Your turn
Sketch the graph of $y = \frac{1}{x} + \frac{256}{3}x^3$ labelling the stationary points.	Sketch the graph of $y = \frac{1}{x} + 27x^3$ labelling the stationary points.



12.10) Sketching gradient functions

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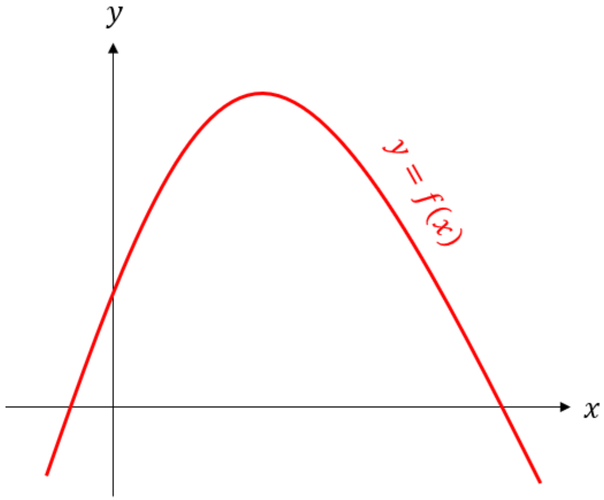
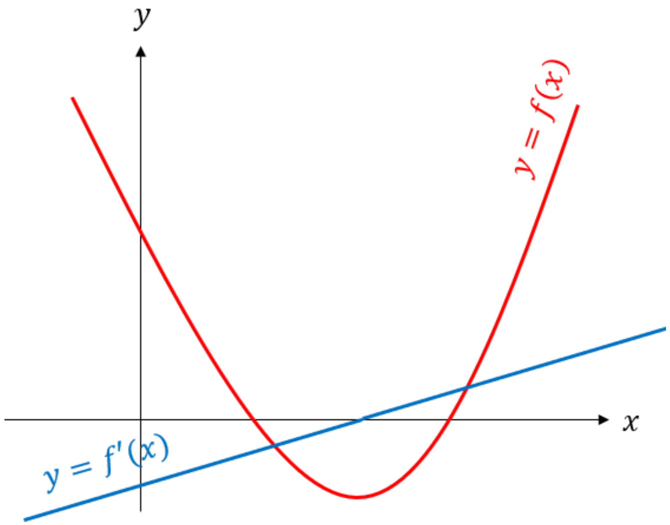
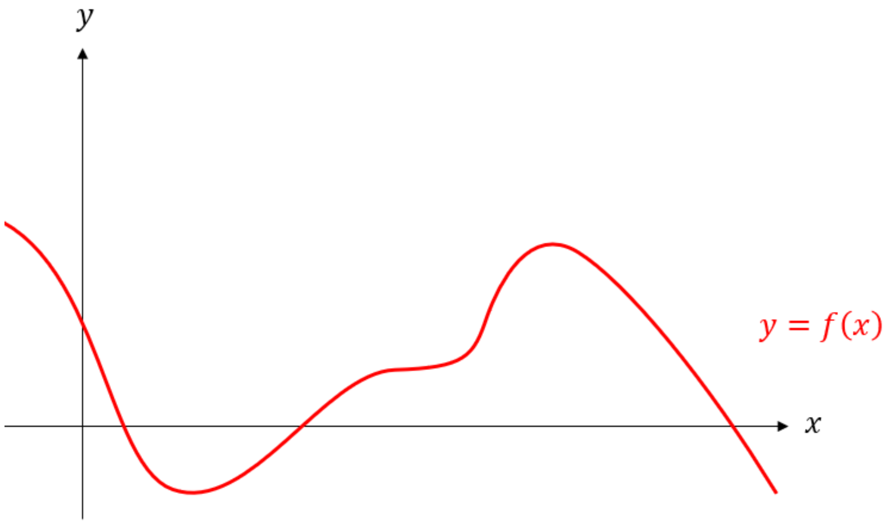
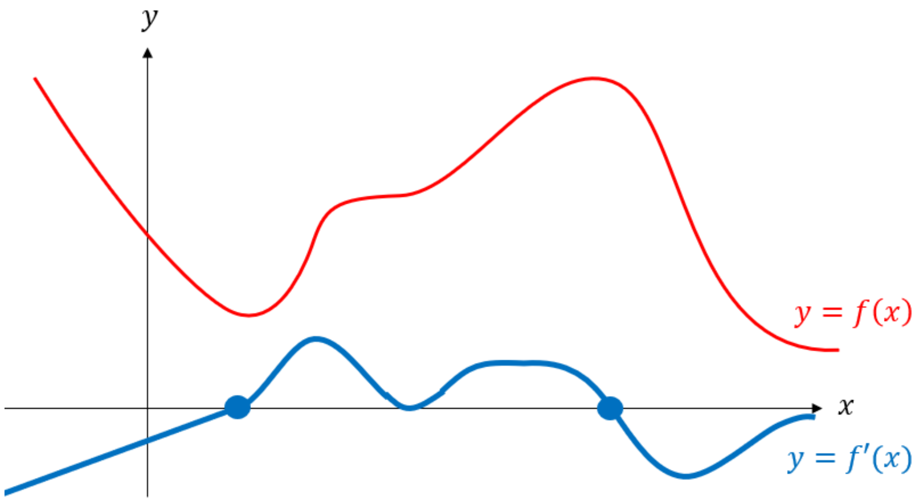
Worked example	Your turn
Sketch $y = f'(x)$ on the same axes	Sketch $y = f'(x)$ on the same axes



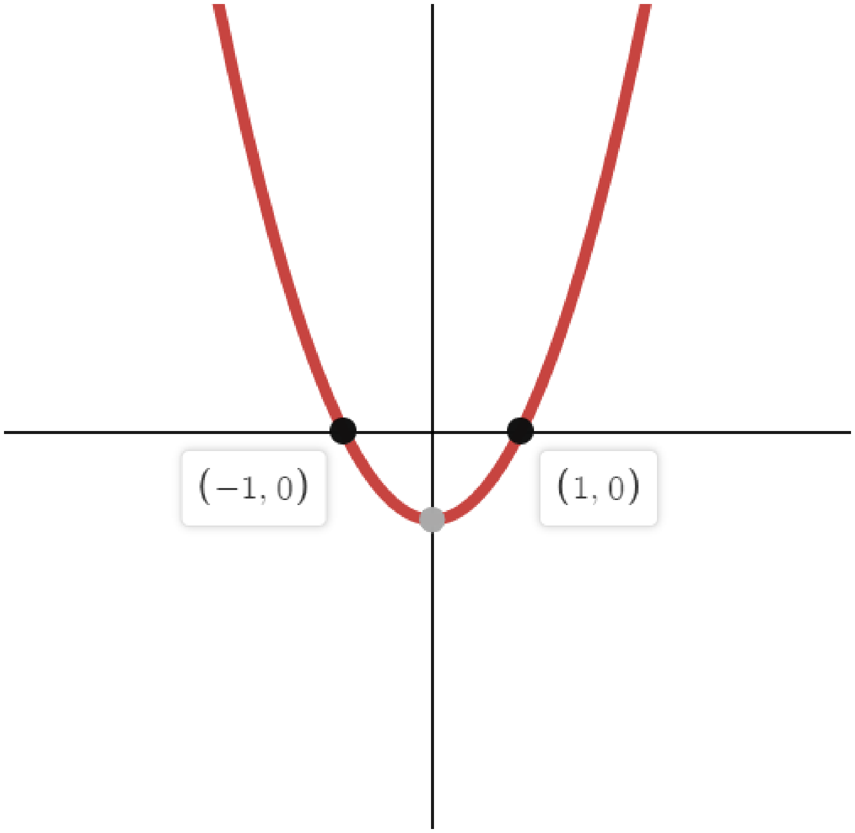


Worked example	Your turn
Sketch $y = f'(x)$ on the same axes	Sketch $y = f'(x)$ on the same axes





Worked example	Your turn
A negative cubic has the equation $y = f(x)$. The curve has stationary points at $(4, 1)$ and $(-1, 0)$ and cuts the x-axis at $(6, 0)$. Sketch the gradient function, $y = f'(x)$, showing the coordinates of any points where the curve cuts or meets the x-axis.	A positive cubic has the equation $y = f(x)$. The curve has stationary points at $(-1, 4)$ and $(1, 0)$ and cuts the x-axis at $(-3, 0)$. Sketch the gradient function, $y = f'(x)$, showing the coordinates of any points where the curve cuts or meets the x-axis.



12.11) Modelling with differentiation

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Worked example	Your turn
Given that the area, $A\ cm^2$, of an expanding circle is related to its radius, $r\ cm$, by the formula $A = \pi r^2$, find the rate of change of area with respect to radius at the instant when the radius is $10\ cm$.	Given that the volume, $V\ cm^3$, of an expanding sphere is related to its radius, $r\ cm$, by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to radius at the instant when the radius is $5\ cm$. $314\ cm^3$ per cm

Worked example	Your turn
A cuboid is to be made with volume 81 cm^3. The cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm. The volume of the cuboid is 81 cm^3. a) Show that the total length, L, of the twelve edges of the cuboid is given by $L = 12x + \frac{162}{x^2}$ b) Given that x can vary, use differentiation to find the maximum or minimum value of L c) Justify that the value of L you have found is a minimum	A cuboid is to be made from 54m² of sheet metal. The cuboid has a horizontal base and no top. The height of the cuboid is x metres. Two of the opposite vertical faces are squares. a) Show that the volume, V m³, of the tank is given by $V = 18x - \frac{2}{3}x^3$. b) Given that x can vary, use differentiation to find the maximum or minimum value of V. c) Justify that the value of V you have found is a maximum a) Shown b) $V = 36$ c) $\frac{d^2V}{dx^2} = -4x\ ;\ x = 3, \frac{d^2V}{dx^2} = -12 < 0$