11.11) Modelling with differential equations

Worked example

The rate of increase of a human population (with population H, where time is t) is proportional to the current population. Form a differential equation, and find its general solution.

Your turn

The rate of increase of a rabbit population (with population P, where time is t) is proportional to the current population. Form a differential equation, and find its general solution.

$$P = Ae^{kt}$$

Worked example

The rate of increase of a population P of microorganisms at time t, in hours, is given by

$$\frac{dP}{dt} = 6P, k > 0$$

Initially the population was of size 4.

a) Find a model for P in the form $P = Ae^{6t}$

b) Find, to the nearest hundred, the size of the population at time $t = 4$

c) Find the time at which the population will be 10000 times its starting value.

d) State one limitation of this model for large values of t

Your turn

The rate of increase of a population P of microorganisms at time t, in hours, is given by

$$\frac{dP}{dt} = 3P, k > 0$$

Initially the population was of size 8.

a) Find a model for P in the form $P = Ae^{3t}$

b) Find, to the nearest hundred, the size of the population at time $t = 2$

c) Find the time at which the population will be 1000 times its starting value.

d) State one limitation of this model for large values of t

a) $P = 8e^{3t}$

b) 3200

c) 2.3 hours $=$ 2 hours 18 minutes

d) The population could not increase in this way forever due to limitations such as available food or space

Worked example

Water in a manufacturing plant is held in a large cylindrical tank of diameter 10m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 64m. 21 minutes later, the height is 27m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 8m.

Your turn

Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m.

a) Shown: $k = \frac{k\sqrt[3]{100\pi h}}{100\pi}$

b) Shown

c) $P = 9, Q = \frac{1}{2}$

d) 16 minutes

Worked example

Liquid is pouring into a container at a constant rate of $40\ cm^3\ s^{-1}$ and is leaking out at a rate proportional to the volume of liquid already in the container.
The volume, $V\ cm^3$, of liquid in the container at time t seconds is satisfied by the differential equation

$$\frac{dV}{dt} = 40 - kV$$

The container is initially empty.
a) By solving the differential equation show that

$$V = A + Be^{-kt}$$

giving the values of A and B in terms of k
b) Given also that $\frac{dV}{dt} = 20$ at $t = 10$, find the volume of liquid in the container 20 seconds after the start.

Your turn

Liquid is pouring into a container at a constant rate of $20\ cm^3\ s^{-1}$ and is leaking out at a rate proportional to the volume of liquid already in the container.
The volume, $V\ cm^3$, of liquid in the container at time t seconds is satisfied by the differential equation

$$\frac{dV}{dt} = 20 - kV$$

The container is initially empty.
a) By solving the differential equation show that

$$V = A + Be^{-kt}$$

giving the values of A and B in terms of k
b) Given also that $\frac{dV}{dt} = 10$ at $t = 5$, find the volume of liquid in the container 10 seconds after the start.

a) $V = \frac{20}{k} - \frac{20}{k}e^{-kt}$

b) $V = \frac{75}{\ln 2} = 108\ cm^3$ (3 sf)

Worked example

A fluid reservoir initially containers 10000 litres of unpolluted fluid.
The reservoir is leaking at a constant rate of 200 litres per hour and it is suspected that contaminated fluid flows into the reservoir at a constant rate of 300 litres per day.
The contaminated fluid contains 4 grams of contaminant in every litre of fluid.
It is assumed that the contaminant instantly disperses throughout the reservoir upon entry.
Given that there are x grams of contaminant in the reservoir after t days,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 1200 - \frac{2x}{100 + t}$$

(b) Hence find the number of grams of contaminant in the tank after 7 days.

(c) Explain how the model could be refined.

Your turn

A storage tank initially containers 1000 litres of pure water.
Liquid is removed from the tank at a constant rate of 30 litres per hour and a chemical solution is added to the tank at a constant rate of 40 litres per hour.
The chemical solution contains 4 grams of copper sulphate per litre of water.
It is assumed that the copper suphate instantly disperses throughout the tank on entry.
Given that there are x grams of copper sulphate in the tank after t hours,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 160 - \frac{3x}{100 + t}$$

(b) Hence find the number of grams of copper sulphate in the tank after 6 hours.

(c) Explain how the model could be refined.

(a) Shown

(b) 882 g (3 sf)

(c) The model could be refined to take into account the fact that the cupper sulphate does not disperse immediately on entering the tank.