11) Integration

11.1) Integrating standard functions

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Worked example	Your turn
By thinking about integration as the reverse of differentiation, find: $$\int x^n \, dx$$ $$\int e^x \, dx$$	By thinking about integration as the reverse of differentiation, find: $$\int \frac{1}{x} \, dx$$ $$\ln\lvert x\rvert + c$$

Worked example	Your turn
By thinking about integration as the reverse of differentiation, find:	By thinking about integration as the reverse of differentiation, find:
$$\int \sin x \, dx$$	$$\int \cos x \, dx$$ $\sin x + c$
$$\int cosec\ x \cot x \, dx$$	$$\int \sec x \tan x \, dx$$ $\sec x + c$
$$\int \sec^2 x \, dx$$	$$\int \text{cosec}^2 x \, dx$$ $-\cot x + c$

Worked example	Your turn
By thinking about integration as the reverse of differentiation, find:	By thinking about integration as the reverse of differentiation, find:
$\int -\sin x \, dx$	$\int -\cos x \, dx$ $-\sin x + c$
$\int -cosec\, x \cot x \, dx$	$\int -\sec x \tan x \, dx$ $-\sec x + c$
$\int -\sec^2 x \, dx$	$\int -\text{cosec}^2 x \, dx$ $\cot x + c$

Worked example	Your turn
Find: $$\int 3\sin x - \frac{4}{x^2} + \sqrt[3]{x}\, dx$$	Find: $$\int 2\cos x + \frac{3}{x} - \sqrt{x}\, dx$$ $$2\sin x + 3\ln\lvert x\rvert - \frac{2}{3}x^{\frac{3}{2}} + c$$

Worked example	Your turn
Find: $$\int \frac{\sin x}{\cos^2 x}\ dx$$	Find: $$\int \frac{\cos x}{\sin^2 x}\ dx$$ $$-cosec\ x + c$$

Worked example	Your turn
Given that $$\int_a^{5a} \frac{3x-1}{x}\,dx = \ln 2,$$ find the exact value of a.	Given that $$\int_a^{3a} \frac{2x+1}{x}\,dx = \ln 12,$$ find the exact value of a. $$a = \frac{1}{4}\ln 4$$

11.2) Integrating $f(ax+b)$

Worked example	Your turn
Find: $$\int (6x+1)^2\,dx$$ $$\int (5x-2)^3\,dx$$ $$\int (4x+3)^4\,dx$$	Find: $$\int (3x-4)^5\,dx$$ $$\frac{1}{18}(3x+4)^6+c$$

Worked example	Your turn
Find: $$\int \frac{1}{2(3x-4)^4}\,dx$$ $$\int \frac{1}{4(2-3x)^3}\,dx$$	Find: $$\int \frac{1}{3(4x-2)^2}\,dx$$ $$-\frac{1}{12(4x-2)}+c$$

Worked example	Your turn
Find:	Find:
$\int (3x-1)^2$	$\int (2x+1)^2$
	$\frac{1}{6}(2x+1)^3+c$
$\int (3x-1)\,dx$	$\int (2x+1)\,dx$
	x^2+x+c
$\int \frac{1}{3x-1}\,dx$	$\int \frac{1}{2x+1}\,dx$
	$\frac{1}{2}\ln\lvert 2x+1\rvert+c$
$\int \frac{1}{(3x-1)^2}\,dx$	$\int \frac{1}{(2x+1)^2}\,dx$
	$-\frac{1}{2(2x+1)}+C$

Worked example	Your turn
Find:	Find:

Worked example

Find:

$$\int \sin(6x+1)\ dx$$

$$\int -\sin\left(\frac{x}{5}-2\right)\ dx$$

$$\int \sin(3-4x)\ dx$$

Your turn

Find:

$$\int -\sin\left(\frac{1}{3}x-4\right)\ dx$$

$$3\,cos\left(\frac{1}{3}x-4\right)+c$$

Worked example	Your turn
Find:	Find:
$$\int \cos(6x+1)\ dx$$	$$\int -\cos(4-3x)\ dx$$
$$\int -\cos\left(\frac{x}{5}-2\right)\ dx$$	$$\frac{1}{3}sin(4-3x)+c$$
$$\int \cos(3-4x)\ dx$$	

Worked example	Your turn
Find:	Find:
$$\int \frac{1}{6x-1}\,dx$$	$$\int \frac{1}{-\frac{1}{3}x+4}\,dx$$
	$$-3\ln\left\lvert -\frac{1}{3}x+4 \right\rvert + c$$
$$\int \frac{1}{\frac{1}{5}x+2}\,dx$$	
$$\int \frac{1}{3-4x}\,dx$$	

Worked example	Your turn
Find: $\int \sec^2(2x-3)\,dx$ $\int 6\sec^2(5-4x)\,dx$	Find: $\int 3\sec^2(2x+1)\,dx$ $\frac{3}{2}\tan(2x+1)+c$

Worked example	Your turn
Find: $$\int \sec 5x \tan 5x \, dx$$ $$\int \sec\frac{x}{4} \tan\frac{x}{4} \, dx$$	Find: $$\int \sec(3x) \tan(3x) \, dx$$ $$\frac{1}{3}\sec(3x) + c$$

Worked example	Your turn
Find: $$\int e^{6x+1}\,dx$$ $$\int e^{\frac{1}{5}x-2}\,dx$$ $$\int e^{4-3x}\,dx$$	Find: $$\int e^{-\frac{1}{3}x-4}\,dx$$ $$-3e^{-\frac{1}{3}x-4}+c$$

Worked example	Your turn
Find: $$\int (e^x - 1)^3 \, dx$$	Find: $$\int (e^x + 1)^2 \, dx$$ $$\frac{1}{2}e^{2x} + 2e^x + x + C$$

11.3) Using trigonometric identities

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Worked example	Your turn
Find: $$\int \sin^2 x \ dx$$	Find: $$\int \cos^2 x \ dx$$ $$\frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

Worked example	Your turn
Find: $$\int \cot^2 x \ dx$$	Find: $$\int \tan^2 x \ dx$$ $$\tan x - x + c$$

Worked example	Your turn
Find: $$\int (\sec x - \tan x)^2 \, dx$$	Find: $$\int (\sec x + \tan x)^2 \, dx$$ $$2\tan x - 2\sec x - x + c$$

Worked example	Your turn
Find: $$\int \sin 5x \cos 5x \ \ dx$$ $$\int \sin \frac{x}{4} \cos \frac{x}{4} \ dx$$	Find: $$\int \sin 3x \cos 3x \ \ dx$$ $$-\frac{1}{12}\cos 6x + c$$

Worked example	Your turn
Find: $$\int (\sin x - \cos x)^2 dx$$	Find: $$\int (\sin x + \cos x)^2 dx$$ $$x - \frac{1}{2}\cos 2x + c$$

Worked example	Your turn
Find: $$\int (\cos 2x + 1)^2 dx$$	Find: $$\int (\cos 2x - 1)^2 dx$$ $$\frac{1}{8}\sin 4x + \frac{3}{2}x - \sin 2x + c$$

Worked example	Your turn
Find: $$\int \frac{(1+\sin x)^2}{\cos^2 x}\,dx$$	Find: $$\int \frac{(1+\cos x)^2}{\sin^2 x}\,dx$$ $$-2\cot x - x - 2\,cosec\,x + c$$

Worked example	Your turn
Find: $$\int \frac{\cos 2x}{\sin^2 x}\,dx$$	Find: $$\int \frac{\cos 2x}{\cos^2 x}\,dx$$ $$2x - \tan x + c$$

Worked example	Your turn
Show that: $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 x \; dx = \frac{\pi}{24} + \frac{2-\sqrt{3}}{8}$$	Show that: $$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x \; dx = \frac{\pi}{48} + \frac{1-\sqrt{2}}{8}$$ $$\frac{\pi}{12}$$

Worked example	Your turn
Find: $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 3x \; dx$$	Find: $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 3x \; dx$$ $$\frac{\pi}{12}$$

Worked example	Your turn
Find: $$\int \sin^4 x \ dx$$	Find: $$\int \cos^4 x \ dx$$ $$\frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c$$

11.4) Reverse chain rule

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Worked example	Your turn
Find: $$\int x^3(x^4-2)^5\,dx$$ $$\int x^2(x^3-5)^4\,dx$$	Find: $$\int x(x^2+5)^3\,dx$$ $$\frac{1}{8}(x^2+5)^4+c$$

Worked example	Your turn
Find: $$\int \sin x \cos^4 x \ dx$$ $$\int \sin^3 x \cos x \, dx$$	Find: $$\int \cos x \sin^2 x \ dx$$ $$\frac{1}{3}\sin^3 x + c$$

Worked example	Your turn
Find:	Find:
$$\int \frac{x^3}{x^4+1}\,dx$$	$$\int \frac{x}{3x^2+2}\,dx$$
	$$\frac{1}{6}\ln\lvert 3x^2+2\rvert + c$$
$$\int \frac{x^2}{4x^3-5}\,dx$$	

Worked example	Your turn
Find:	Find:
$$\int \frac{4x^3}{x^4+1}\,dx$$	$$\int \frac{6x}{3x^2+2}\,dx$$
	$$\ln\lvert 3x^2+2\rvert + c$$
$$\int \frac{12x^2}{4x^3-5}\,dx$$	

Worked example	Your turn
Find: $$\int \frac{x}{(x^2-3)^5}\,dx$$ $$\int \frac{x^2}{(x^3-2)^4}\,dx$$	Find: $$\int \frac{x}{(x^2+5)^3}\,dx$$ $$-\frac{1}{4}(x^2+5)^{-2}+c$$

Worked example	Your turn
Find: $$\int x^3\, e^{x^4+5}\, dx$$ $$\int x^2\, e^{x^3-4}\, dx$$	Find: $$\int x\, e^{x^2+1}\, dx$$ $$\frac{1}{2}e^{x^2+1} + c$$

Worked example	Your turn
Find:	Find:

Worked example

Find:

$$\int \frac{3x^2 + 5}{\sqrt{x^3 + 5x - 2}}\, dx$$

$$\int \frac{12x^3 - 45x^2}{\sqrt{x^4 - 5x^3 + 1}}\, dx$$

Your turn

Find:

$$\int \frac{2x + 1}{\sqrt{x^2 + x - 3}}\, dx$$

$$2\sqrt{x^2 + x - 3} + c$$

Worked example	Your turn
Find: $$\int \frac{\sin x}{\cos x + 5}\,dx$$ $$\int \frac{\sec^2 x}{\tan x - 3}\,dx$$	Find: $$\int \frac{\cos x}{\sin x + 2}\,dx$$ $$ln\|\sin x + 2\| + c$$

Worked example	Your turn
Find: $$\int \sin x \; e^{\cos x}\,dx$$ $$\int \sec^2 x \; e^{\tan x}\,dx$$	Find: $$\int \cos x \; e^{\sin x}\,dx$$ $$e^{\sin x} + c$$

Worked example	Your turn
Find: $$\int \sin x\,(\cos x - 1)^5\,dx$$ $$\int \sec^2 x\,(\tan x + 3)^6\,dx$$	Find: $$\int \cos x\,(\sin x - 5)^7\,dx$$ $$\frac{1}{8}(\sin x - 5)^8 + c$$

Worked example	Your turn
Find: $$\int \sec^2 x \tan x \; dx$$ $$\int 2\sec^3 x \tan x \; dx$$	Find: $$\int 3\sec^4 x \tan x \; dx$$ $$\frac{3}{4}\sec^4 x + c$$

Worked example	Your turn
Find: $$\int \frac{cosec^2 x}{(3-\cot x)^4}\,dx$$ $$\int \frac{sec^2 x}{(3+\tan x)^2}\,dx$$	Find: $$\int \frac{cosec^2 x}{(2+\cot x)^3}\,dx$$ $$\frac{1}{2}(2+\cot x)^{-2}+c$$

Worked example	Your turn
Find: $$\int \frac{\sin 4x}{\cos 4x + 2}\,dx$$ $$\int \frac{\sec^2 3x}{\tan 3x - 5}\,dx$$	Find: $$\int \frac{\cos 2x}{\sin 2x + 3}\,dx$$ $$\frac{1}{2} ln\lvert\sin 2x + 3\rvert + c$$

Worked example	Your turn
Find: $$\int \tan x \, dx$$	Find: $$\int \cot x \, dx$$ $$\ln \lvert \sin x \rvert + c$$

11.5) Integration by substitution

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Worked example

Find:

$$\int x(2x-5)^9\ dx$$

using the substitution $u = 2x - 5$

Your turn

Find:

$$\int x(5x-2)^8\ dx$$

using the substitution $u = 5x - 2$

$$\frac{(5x-2)^{10}}{250} + \frac{2(5x-2)^9}{225} + c$$

Worked example	Your turn
Find: $$\int x\sqrt{5x+2}\ dx$$ using the substitution $u = 5x + 2$	Find: $$\int x\sqrt{2x+5}\ dx$$ using the substitution $u = 2x + 5$ $$\frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + c$$

Worked example	Your turn
Find: $$\int x\sqrt{5x+2}\ dx$$ using the substitution $u^2 = 5x + 2$	Find: $$\int x\sqrt{2x+5}\ dx$$ using the substitution $u^2 = 2x + 5$ $$\frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + c$$

Worked example	Your turn
Find: $$\int \cos x \sin x\,(2+\sin x)^4\ dx$$ using the substitution $u = \sin x + 2$	Find: $$\int \cos x \sin x\,(1+\sin x)^3\ dx$$ using the substitution $u = \sin x + 1$ $$\frac{1}{5}(\sin x + 1)^5 - \frac{1}{4}(\sin x + 1)^4 + c$$

Worked example	Your turn
Find: $$\int_0^{\frac{\pi}{2}} \cos x \sin x \,(2+\sin x)^4 \, dx$$ using the substitution $u = \sin x + 2$	Find: $$\int_0^{\frac{\pi}{2}} \cos x \sin x \,(1+\sin x)^3 \, dx$$ using the substitution $u = \sin x + 1$ $$\frac{49}{20}$$

Worked example	Your turn
Find: $$\int \frac{3\sin 2x}{2+\sin x}\,dx$$ using the substitution $u = 2 + \sin x$	Find: $$\int \frac{2\sin 2x}{1+\cos x}\,dx$$ using the substitution $u = 1 + \cos x$ $$4\ln\lvert 1+\cos x\rvert - 4\cos x + c$$

Worked example	Your turn
Calculate: $$\int_0^{\frac{\pi}{2}} \sin x\sqrt{2+\cos x}\ dx$$ using the substitution $u = \cos x + 2$	Calculate: $$\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin x}\ dx$$ using the substitution $u = \sin x + 1$ $$\frac{2}{3}\left(2\sqrt{2}-1\right)$$

Worked example	Your turn
Use the substitution $u = \sqrt{x} - 1$ to evaluate: $$\int_{36}^{49} \frac{1}{\sqrt{x} - 1}\, dx$$	Use the substitution $u = 1 + \sqrt{x}$ to evaluate: $$\int_{16}^{25} \frac{1}{1 + \sqrt{x}}\, dx$$ $$2 + 2\ln\left\lvert\frac{5}{6}\right\rvert$$

Worked example	Your turn
A finite region is bounded by the curve with equation $y = x^3 \ln(x^2 + 3)$, the x-axis and the lines $x = 0$ and $x = \sqrt{5}$. Use the substitution $u = x^2 + 3$ to show that the area of R is $\frac{1}{2}\int_3^8 (u - 3) \ln u \, du$	A finite region is bounded by the curve with equation $y = x^3 \ln(x^2 + 2)$, the x-axis and the lines $x = 0$ and $x = \sqrt{2}$. Use the substitution $u = x^2 + 2$ to show that the area of R is $\frac{1}{2}\int_2^4 (u - 2) \ln u \, du$ Shown

Worked example	Your turn
Using integration by substitution, prove that: $$-\int \frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$$	Using integration by substitution, prove that: $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$$ Shown

Worked example

Use the substitution $u = \cos x$ to evaluate

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \, dx$$

Your turn

Use the substitution $u = \sin x$ to evaluate

$$\int_0^{\frac{\pi}{6}} \sin^2 x \cos^3 x \, dx$$

$$\frac{17}{480}$$

Worked example	Your turn
Use the substitution $x = \cos u$ to evaluate $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2}\,dx$$	Use the substitution $x = \sin u$ to evaluate $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2}\,dx$$ $$\frac{2\pi + 3\sqrt{3}}{96}$$

11.6) Integration by parts

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Worked example	Your turn
Find: $$\int x \sin x \; dx$$	Find: $$\int x \cos x \; dx$$ $$x \sin x + \cos x + c$$

Worked example	Your turn
Find: $$\int x^2 \cos x \ dx$$	Find: $$\int x^2 \sin x \ dx$$ $$2x \sin x - x^2 \cos x + 2 \cos x + c$$

Worked example	Your turn
Find: $$\int x^2 e^{-x}\, dx$$	Find: $$\int x^2 e^{x}\, dx$$ $$x^2e^x - 2xe^x + 2e^x + c$$

Worked example	Your turn
Find: $$\int x^3 \ln x \ dx$$	Find: $$\int x^2 \ln x \ dx$$ $$\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$$

Worked example	Your turn
Evaluate: $$\int_1^4 \ln x \, dx$$	Evaluate: $$\int_1^2 \ln x \, dx$$ $2\ln 2 - 1$

Worked example	Your turn
Find: $$\int e^x \cos x \; dx$$	Find: $$\int e^x \sin x \; dx$$ $$-\frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + c$$

Worked example	Your turn
Evaluate: $$\int_0^{\frac{\pi}{2}} x \cos x \ dx$$	Evaluate: $$\int_0^{\frac{\pi}{2}} x \sin x \ dx$$ 1

11.7) Partial fractions

Worked example	Your turn
Find: $$\int \frac{x+5}{(x-1)(x+2)}\,dx$$	Find: $$\int \frac{x-5}{(x+1)(x-2)}\,dx$$ $$\ln\left|\frac{(x+1)^2}{x-2}\right| + c$$

Worked example	Your turn
Find: $$\int \frac{3x + 15 - 4x^2}{(2x+1)(x-2)^2}\,dx$$	Find: $$\int \frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2}\,dx$$ $$\ln\left|(2x+1)(x-2)^3\right| + \frac{1}{x-2} + c$$

Worked example	Your turn
Evaluate: $$\int_0^2 \frac{8x^2+34x+20}{(2x+1)(x+1)(x+3)}$$	Evaluate: $$\int_0^2 \frac{4-2x}{(2x+1)(x+1)(x+3)}$$ $$\ln\left(\frac{125}{81}\right)$$

Worked example	Your turn
Find: $$\int \frac{4}{x^2 - 4}\, dx$$	Find: $$\int \frac{2}{x^2 - 1}\, dx$$ $$ln\left|\frac{x-1}{x+1}\right| + c$$

Worked example	Your turn
Find: $$\int \frac{x^2}{x-1}\,dx$$	Find: $$\int \frac{x^2}{x+1}\,dx$$ $$\frac{1}{2}x^2 - x + \ln\lvert x+1\rvert + c$$

Worked example	Your turn
Find: $$\int \frac{x}{x-2}\,dx$$	Find: $$\int \frac{x}{x-1}\,dx$$ $$x + ln\lvert x-1\rvert + c$$
$$\int \frac{x-2}{x}\,dx$$	$$\int \frac{x-1}{x}\,dx$$ $$x - \ln\lvert x\rvert + c$$

Worked example	Your turn
Find: $$\int \frac{x^3 - 2}{x - 1}\, dx$$	Find: $$\int \frac{x^3 + 2}{x + 1}\, dx$$ $$\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \ln\lvert x + 1\rvert + c$$

Worked example	Your turn
Find: $$\int \frac{4x^2 - 2x - 18}{4x^2 - 9}\,dx$$	Find: $$\int \frac{9x^2 - 3x + 2}{9x^2 - 4}\,dx$$ $$x + \frac{1}{3}\ln\left\|\frac{3x - 2}{(3x + 2)^2}\right\| + c$$

11.8) Finding areas

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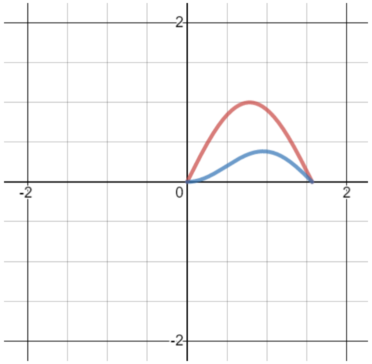
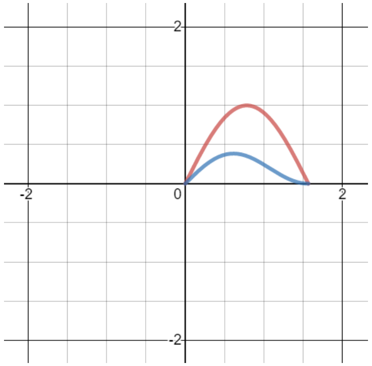
Worked example	Your turn
A finite region is bound by the curve $y = \frac{3}{\sqrt{9+4x}}$, the x-axis, and the lines $x = 0$ and $x = 4$. Use integration to find the area of the region.	A finite region is bound by the curve $y = \frac{9}{\sqrt{4+3x}}$, the x-axis, and the lines $x = 0$ and $x = 4$. Use integration to find the area of the region. 12

Worked example	Your turn
A finite region is bound between the curves $y = \sin 2x$ and $y = cos\, x \sin^2 x$ where $0 \le x \le \frac{\pi}{2}$. Use integration to find the area of the region. 	A finite region is bound between the curves $y = \sin 2x$ and $y = \sin x \cos^2 x$ where $0 \le x \le \frac{\pi}{2}$. Use integration to find the area of the region. $\frac{2}{3}$ 

11.8+) Finding areas: Areas under parametric curves

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Worked example	Your turn
Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t - 1$, the x-axis, and the lines $x = 0$ and $x = 5$.	Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t + 1$, the x-axis, and the lines $x = 0$ and $x = 3$. $2\sqrt{3} + 3$

Worked example	Your turn
The curve C has parametric equations $x = t(2+t), \quad y = \frac{1}{2+t}, \quad t \geq 0$ Find the exact area of the region, bounded by C, the x-axis and the lines $x = 0$ and $x = 8$.	The curve C has parametric equations $x = t(1+t), \quad y = \frac{1}{1+t}, \quad t \geq 0$ Find the exact area of the region, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$. $2 - \ln 2$

Worked example	Your turn
The curve C has parametric equations $$x = 1 - \frac{1}{4}t, \qquad y = 4^t - 1, \qquad t \geq 0$$ A finite region is bounded by the curve C, the x-axis and the line $x = -1$. Find the exact area of this region.	The curve C has parametric equations $$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1, \qquad t \geq 0$$ A finite region is bounded by the curve C, the x-axis and the line $x = -1$. Find the exact area of this region. $$\frac{15}{2\ln 2} - 2$$

11.9) The trapezium rule

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Worked example	Your turn
Using the trapezium rule, approximate the region bounded between $x = 1$, $x = 3$, the x-axis and the curve $y = x^2$, using 8 strips.	Using the trapezium rule, approximate the region bounded between $x = 1$, $x = 3$, the x-axis and the curve $y = x^2$, using 4 strips. 8.75

Worked example	Your turn
$$I = \int_0^{\frac{\pi}{3}} \sec x \ dx$$ Use the trapezium rule with two strips to estimate I.	$$I = \int_0^{\frac{\pi}{3}} \sec x \ dx$$ Use the trapezium rule with four strips to estimate I. 1.34 (2 dp)

Worked example	Your turn
$$I=\int_0^{\frac{\pi}{2}}\sin x\ dx$$	$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\ dx$$

Worked example

$$I=\int_0^{\frac{\pi}{2}}\sin x\ dx$$

a) Use the trapezium rule with four strips to estimate I
b) State, with a reason, whether your approximation is an underestimate or an overestimate
c) Find the percentage error of your estimate to the exact value of I
d) Give one way the trapezium rule can be used to give a more accurate approximation

Your turn

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\ dx$$

a) Use the trapezium rule with four strips to estimate I
b) State, with a reason, whether your approximation is an underestimate or an overestimate
c) Find the percentage error of your estimate to the exact value of I
d) Give one way the trapezium rule can be used to give a more accurate approximation

a) 1.896 (3 dp)
b) Underestimate. The graph of $y=\cos x$ is convex in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$
c) 5.19% (3 sf)
d) Increase number of trapezia, decrease h, increase n etc.

11.10) Solving differential equations

Worked example	Your turn
Find the general solution to: $$\frac{dy}{dx} = xy - y$$	Find the general solution to: $$\frac{dy}{dx} = xy + y$$ $$y = Ae^{\frac{1}{2}x^2+x}$$

Worked example	Your turn
Find the general solution to: $$(1-x^2)\frac{dy}{dx} = x\cot y$$	Find the general solution to: $$(1+x^2)\frac{dy}{dx} = x\tan y$$ $$y = \arcsin\left(k\sqrt{1+x^2}\right)$$

Worked example	Your turn
Find the particular solution to: $$\frac{dy}{dx} = -\frac{3(y+2)}{(2x-1)(x-2)}$$ given that $x = 4$ when $y = 5$	Find the particular solution to: $$\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$$ given that $x = 1$ when $y = 4$ $$y = 3 + \frac{3}{2x+1}$$

Worked example	Your turn
Find the particular solution to: $$\frac{dy}{dx} = -\frac{5}{y\sin^2 x}$$ given that $y = 4$ at $x = \frac{\pi}{4}$	Find the particular solution to: $$\frac{dy}{dx} = -\frac{3}{y\cos^2 x}$$ given that $y = 2$ at $x = \frac{\pi}{4}$ $$y^2 = -6\tan x + 10$$

Worked example	Your turn
Find the particular solution to: $$\frac{dy}{dx} = xy\cos x$$ given that $y = 1$ at $x = \frac{\pi}{2}$	Find the particular solution to: $$\frac{dy}{dx} = xy\sin x$$ given that $y = 1$ at $x = \frac{\pi}{2}$ $$\ln\lvert y\rvert = \sin x - x\cos x - 1$$

11.11) Modelling with differential equations

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Worked example	Your turn
The rate of increase of a human population (with population H, where time is t) is proportional to the current population. Form a differential equation, and find its general solution.	The rate of increase of a rabbit population (with population P, where time is t) is proportional to the current population. Form a differential equation, and find its general solution. $P = Ae^{kt}$

Worked example	Your turn
The rate of increase of a population P of microorganisms at time t, in hours, is given by $$\frac{dP}{dt} = 6P, k > 0$$ Initially the population was of size 4. a) Find a model for P in the form $P = Ae^{6t}$ b) Find, to the nearest hundred, the size of the population at time $t = 4$ c) Find the time at which the population will be 10000 times its starting value. d) State one limitation of this model for large values of t	The rate of increase of a population P of microorganisms at time t, in hours, is given by $$\frac{dP}{dt} = 3P, k > 0$$ Initially the population was of size 8. a) Find a model for P in the form $P = Ae^{3t}$ b) Find, to the nearest hundred, the size of the population at time $t = 2$ c) Find the time at which the population will be 1000 times its starting value. d) State one limitation of this model for large values of t a) $P = 8e^{3t}$ b) 3200 c) 2.3 hours $=$ 2 hours 18 minutes d) The population could not increase in this way forever due to limitations such as available food or space

Worked example	Your turn
Water in a manufacturing plant is held in a large cylindrical tank of diameter 10m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume. (a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k. (b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants. Initially the height of the water is 64m. 21 minutes later, the height is 27m. (c) Find the values of the constants P and Q. (d) Find the time in minutes when the water is at a depth of 8m.	Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume. (a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k. (b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants. Initially the height of the water is 27m. 10 minutes later, the height is 8m. (c) Find the values of the constants P and Q. (d) Find the time in minutes when the water is at a depth of 1m. a) Shown: $k = \frac{k\sqrt[3]{100\pi h}}{100\pi}$ b) Shown c) $P = 9, Q = \frac{1}{2}$ d) 16 minutes

Worked example

Liquid is pouring into a container at a constant rate of $40\ cm^3\ s^{-1}$ and is leaking out at a rate proportional to the volume of liquid already in the container.
The volume, $V\ cm^3$, of liquid in the container at time t seconds is satisfied by the differential equation

$$\frac{dV}{dt} = 40 - kV$$

The container is initially empty.
a) By solving the differential equation show that

$$V = A + Be^{-kt}$$

giving the values of A and B in terms of k
b) Given also that $\frac{dV}{dt} = 20$ at $t = 10$, find the volume of liquid in the container 20 seconds after the start.

Your turn

Liquid is pouring into a container at a constant rate of $20\ cm^3\ s^{-1}$ and is leaking out at a rate proportional to the volume of liquid already in the container.
The volume, $V\ cm^3$, of liquid in the container at time t seconds is satisfied by the differential equation

$$\frac{dV}{dt} = 20 - kV$$

The container is initially empty.
a) By solving the differential equation show that

$$V = A + Be^{-kt}$$

giving the values of A and B in terms of k
b) Given also that $\frac{dV}{dt} = 10$ at $t = 5$, find the volume of liquid in the container 10 seconds after the start.

a) $V = \frac{20}{k} - \frac{20}{k}e^{-kt}$

b) $V = \frac{75}{\ln 2} = 108\ cm^3$ (3 sf)

Worked example

A fluid reservoir initially containers 10000 litres of unpolluted fluid.
The reservoir is leaking at a constant rate of 200 litres per hour and it is suspected that contaminated fluid flows into the reservoir at a constant rate of 300 litres per day.
The contaminated fluid contains 4 grams of contaminant in every litre of fluid.
It is assumed that the contaminant instantly disperses throughout the reservoir upon entry.
Given that there are x grams of contaminant in the reservoir after t days,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 1200 - \frac{2x}{100 + t}$$

(b) Hence find the number of grams of contaminant in the tank after 7 days.

(c) Explain how the model could be refined.

Your turn

A storage tank initially containers 1000 litres of pure water.
Liquid is removed from the tank at a constant rate of 30 litres per hour and a chemical solution is added to the tank at a constant rate of 40 litres per hour.
The chemical solution contains 4 grams of copper sulphate per litre of water.
It is assumed that the copper suphate instantly disperses throughout the tank on entry.
Given that there are x grams of copper sulphate in the tank after t hours,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 160 - \frac{3x}{100 + t}$$

(b) Hence find the number of grams of copper sulphate in the tank after 6 hours.

(c) Explain how the model could be refined.

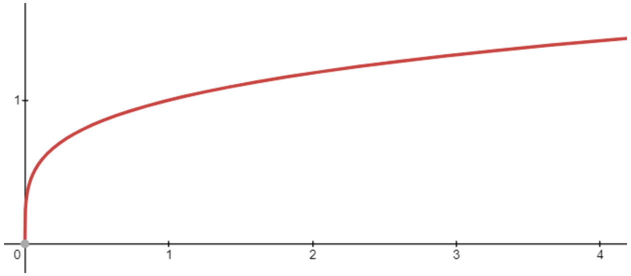
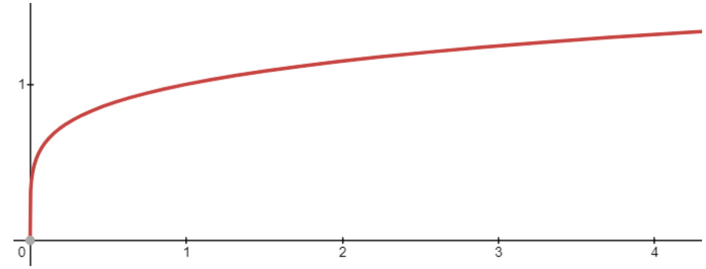
(a) Shown

(b) 882 g (3 sf)

(c) The model could be refined to take into account the fact that the cupper sulphate does not disperse immediately on entering the tank.

11.12) Integration as the limit of a sum

Chapter CONTENTS

Worked example	Your turn
The diagram shows a sketch of the curve with equation $y = \sqrt[4]{x}, x > 0$.	The diagram shows a sketch of the curve with equation $y = \sqrt[5]{x}, x > 0$.
	
The area under the curve may be thought of as a series of thin strips of height y and width δx.	The area under the curve may be thought of as a series of thin strips of height y and width δx.
Calculate to 4 significant figures:	Calculate to 4 significant figures:
$$\lim_{\delta x \to 0} \sum_{2}^{3} \sqrt[4]{x}\,\delta x$$	$$\lim_{\delta x \to 0} \sum_{3}^{4} \sqrt[5]{x}\,\delta x$$
	1.284

Worked example	Your turn
Calculate to four significant figures: $$\lim_{\delta x \to 0} \sum_{5}^{6} \cos x \, \delta x$$	Calculate to four significant figures: $$\lim_{\delta x \to 0} \sum_{1}^{2} \sin x \, \delta x$$ 0.9564