10.3) The Newton-Raphson method

Worked example	Your turn
Using the Newton-Raphson process, state the recurrence relation for the following functions:	Using the Newton-Raphson process, state the recurrence relation for the following functions:
$f(x) = x^4 - 3$	$f(x) = x^3 - 2$ $x_{n+1} = x_n - \frac{(x_n)^3 - 2}{3(x_n)^2}$
$g(x) = \sec x$	$g(x) = \tan x$ $x_{n+1} = x_n - \frac{\tan x_n}{\sec^2 x_n} = x_n - \frac{1}{2}\sin(2x_n)$
$h(x) = x^2 + x + 3$	$h(x) = x^2 - x - 1$ $x_{n+1} = x_n - \frac{(x_n)^2 - x_n - 1}{2x_n - 1}$

Worked example	Your turn
Using three iterations of the Newton-Raphson process, starting with $x_0 = 0.5$, solve the equation $$x = \sin x$$	Using three iterations of the Newton-Raphson process, starting with $x_0 = 0.5$, solve the equation $$x = \cos x$$ Let $f(x) = x - \cos x$ $f'(x) = 1 + \sin x$ $x_{n+1} = x_n - \frac{x_n - cos(x_n)}{1 + sin(x_n)}, x_0 = 0.5$ $x_1 = 0.5 - \frac{0.5 - cos(0.5)}{1 + sin(0.5)} = 0.7552224 \ldots$ $x_2 = 0.7391412$ $x_3 = 0.7390851$ $x = 0.739$ (3 dp)

Worked example	Your turn
$$f(x) = \frac{1}{3}x^4 - x^2 + 3x - 1$$ The equation $f(x) = 0$ has a root α in the interval $[-2, -3]$ Taking -2.5 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 2 decimal places.	$$g(x) = \frac{1}{2}x^4 - x^3 + x - 3$$ The equation $g(x) = 0$ has a root β in the interval $[-2, -1]$ Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $g(x)$ to obtain a second approximation to β. Give your answer to 2 decimal places. $$x_{n+1} = x_n - \frac{\frac{1}{2}(x_n)^4 - (x_n)^3 + x_n - 3}{2(x_n)^3 - 3(x_n)^2 + 1}$$ $$\beta_1 = -1.5 - \frac{1.40625}{-12.5} =- -1.3875 \ldots$$ $$= 1.39 \text{ (2 dp)}$$

Worked example	Your turn
$f(x) = 11x^2 - \frac{3}{x^2}$ The equation $f(x) = 0$ has a root α in the interval $[0, 1]$ Taking 0.4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.	$g(x) = 3x^2 - \frac{11}{x^2}$ The equation $g(x) = 0$ has a root β in the interval $[1, 2]$ Taking 1.4 as a first approximation to β, apply the Newton-Raphson process once to $g(x)$ to obtain a second approximation to β. Give your answer to 3 decimal places. $x_{n+1} = x_n - \dfrac{3(x_n)^2 - \dfrac{11}{(x_n)^2}}{6x_n + \dfrac{22}{(x_n)^3}}$ $\beta_1 = 1.4 - \frac{0.2677...}{16.4174...} = 1.38369...$ $= 1.384$ (3 dp)

Worked example	Your turn
$f(x) = x^2 - 5x + 8$ State why $x_0 = 2.5$ is not suitable to use as a first approximation to the roots of $f(x)$ when applying the Newton-Raphson method.	$f(x) = x^2 + 7x + 8$ State why $x_0 = -3.5$ is not suitable to use as a first approximation to the roots of $f(x)$ when applying the Newton-Raphson method. $f'(x) = 2x + 7 = 0$ -> $x = -3.5$ Turning point at $x = -3.5$ $f'(-3.5) = 0$ You cannot divide by 0 in the Newton-Raphson method. Also the tangent to $y = f(x)$ at $x = -3.5$ would be horizontal, and therefore never intersect the x-axis.