10.2) Iteration

Worked example	Your turn
$f(x) = x^2 - 5x - 3$	$g(x) = x^2 - 6x + 2$
a) Show that $f(x) = 0$ can be written as:	a) Show that $g(x) = 0$ can be written as:
i) $x = \frac{x^2-3}{5}$ ii) $x = \sqrt{5x+3}$ iii) $x = 5 + \frac{3}{x}$	i) $x = \frac{x^2+2}{6}$ ii) $x = \sqrt{6x-2}$ iii) $x = 6 - \frac{2}{x}$
	a) Shown

Worked example	Your turn
$f(x) = x^2 - 5x - 3$	$g(x) = x^2 - 6x + 2$
a) Show that $f(x) = 0$ can be written as:	a) Show that $g(x) = 0$ can be written as:
i) $x = \frac{x^2-3}{5}$ ii) $x = \sqrt{5x+3}$ iii) $x = 5 + \frac{3}{x}$	i) $x = \frac{x^2+2}{6}$ ii) $x = \sqrt{6x-2}$ iii) $x = 6 - \frac{2}{x}$
b) Starting with $x_0 = 3$ use each iterative formula to find a root of the equation $f(x) = 0$, rounding your answers to 3 decimal places	b) Starting with $x_0 = 4$ use each iterative formula to find a root of the equation $g(x) = 0$, rounding your answers to 3 decimal places
	b) i) $x = 0.354$ (3 dp) ii) $x = 5.646$ (3 dp) iii) $x = 5.646$ (3 dp)

Worked example

$$f(x) = e^{x-2} + x - 5$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(5 - x) + 2, \qquad x < 5$$

Your turn

$$f(x) = e^{x-1} + x - 6$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(6 - x) + 1, \qquad x < 6$$

a) Shown

Worked example	Your turn
$f(x) = e^{x-2} + x - 5$	$f(x) = e^{x-1} + x - 6$

Worked example

$$f(x) = e^{x-2} + x - 5$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(5 - x) + 2, \qquad x < 5$$

The root of $f(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(5 - x_n) + 2, \qquad x_0 = 3$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four

decimal places.

Your turn

$$f(x) = e^{x-1} + x - 6$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(6 - x) + 1, \qquad x < 6$$

The root of $f(x) = 0$ is β.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for β

b) Calculate the values of x_1, x_2 and x_3 to four

decimal places.

b)

$x_0 = 2$

$x_1 = \ln(6 - 2) + 1$

$= 2.3863$

$x_2 = 2.2847 \ldots$

$x_3 = 2.3125$

Worked example

$$f(x) = e^{x-2} + x - 5$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(5 - x) + 2, \qquad x < 5$$

The root of $f(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(5 - x_n) + 2, \qquad x_0 = 3$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four decimal places.

c) By choosing a suitable interval, show that $\alpha = 2.792$ correct to 3 decimal places.

Your turn

$$f(x) = e^{x-1} + x - 6$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(6 - x) + 1, \qquad x < 6$$

The root of $f(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four decimal places.

c) By choosing a suitable interval, show that $\beta = 2.307$ correct to 3 decimal places.

c)

$f(2.3065) = -0.00027 \ldots < 0$

$f(2.3075) = 0.0044 \ldots > 0$

Sign change and $g(x)$ continuous in the interval $[2.3065, 2.3075]$

$\therefore 2.3065 < \beta < 2.3075$

$\therefore \beta = 2.307$ (3 dp)

Worked example	Your turn
$f(x) = x^3 - 5x^2 - 3x + 2$ (a) Show that the equation $f(x) = 0$ has a root in the interval $5 < x < 6$.	$g(x) = x^3 - 3x^2 - 2x + 5$ (a) Show that the equation $g(x) = 0$ has a root in the interval $3 < x < 4$. (a) $g(3) = -1 < 0$ $g(4) = 13 > 0$ Change of sign and $g(x)$ continuous in the interval $[3, 4]$ $\therefore$ root in the interval $[3, 4]$

Worked example

$$f(x) = x^3 - 5x^2 - 3x + 2$$

(a) Show that the equation $f(x) = 0$ has a root in the interval $5 < x < 6$.

(b) Use the iterative formula

$$x_{n+1} = \sqrt{\frac{x_n^3 - 3x_n + 2}{5}}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking:

(i) $x_0 = 0.5$ (ii) $x_0 = 6$

Your turn

$$g(x) = x^3 - 3x^2 - 2x + 5$$

(a) Show that the equation $g(x) = 0$ has a root in the interval $3 < x < 4$.

(b) Use the iterative formula

$$x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking:

(i) $x_0 = 1.5$ (ii) $x_0 = 4$

(b)

i)

$$x_1 = \sqrt{\frac{1.5^3 - 2(1.5) + 5}{3}} = 1.3385 \ldots$$

$x_2 = 1.2544 \ldots$

$x_3 = 1.2200 \ldots$

Convergent as the change in the root on each iteration is decreasing. The iterative method will find the root.

ii)

$$x_1 = \sqrt{\frac{4^3 - 2(4) + 5}{3}} = 4.5092 \ldots$$

$x_2 = 5.4058$

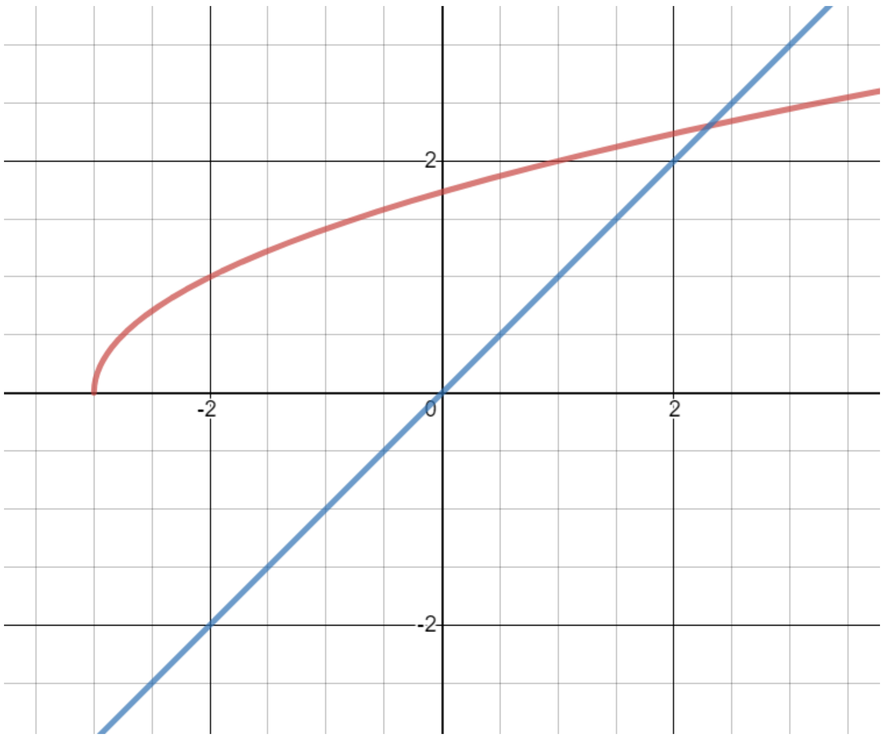
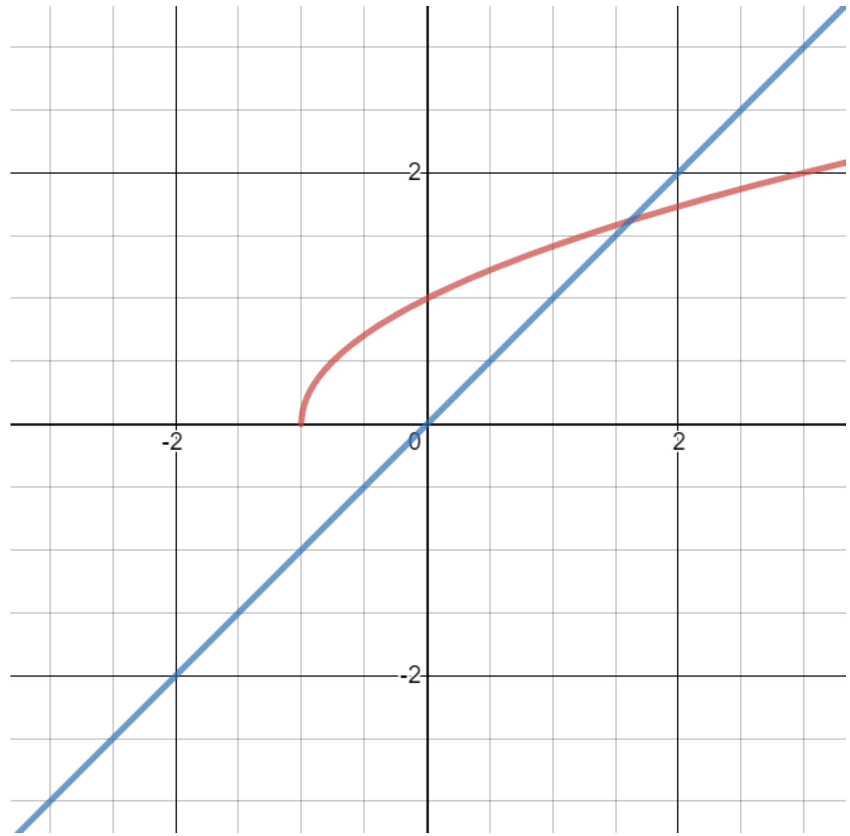
$x_3 = 7.1219 \ldots$

Divergent as the change in the root on each iteration is increasing. The iterative method has failed to find the root.

Worked example	Your turn
$f(x) = x^3 + 4x^2 + 3x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$	$g(x) = x^3 + 3x^2 + 4x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{4(3-x)}{3+x}}, x \neq -3$ (a) Shown

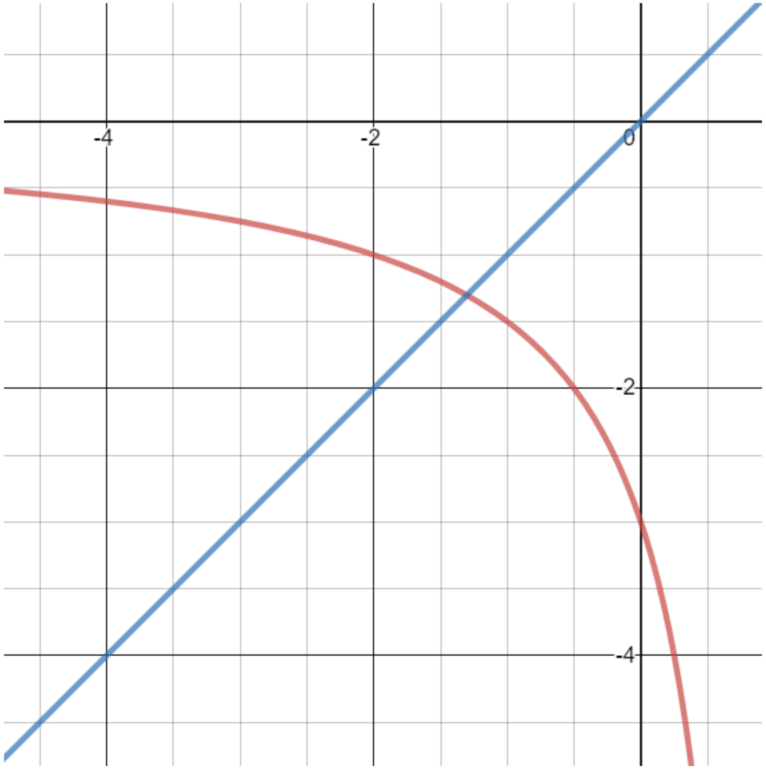
Worked example	Your turn
$f(x) = x^3 + 4x^2 + 3x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$ The equation $f(x) = 0$ has a single root between 1 and 2. (b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places.	$g(x) = x^3 + 3x^2 + 4x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{4(3-x)}{3+x}}, x \neq -3$ The equation $g(x) = 0$ has a single root between 1 and 2. (b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places. (b) $x_1 = \sqrt{\frac{4(3-1)}{3+1}} = 1.41$ $x_2 = 1.20$ $x_3 = 1.31$

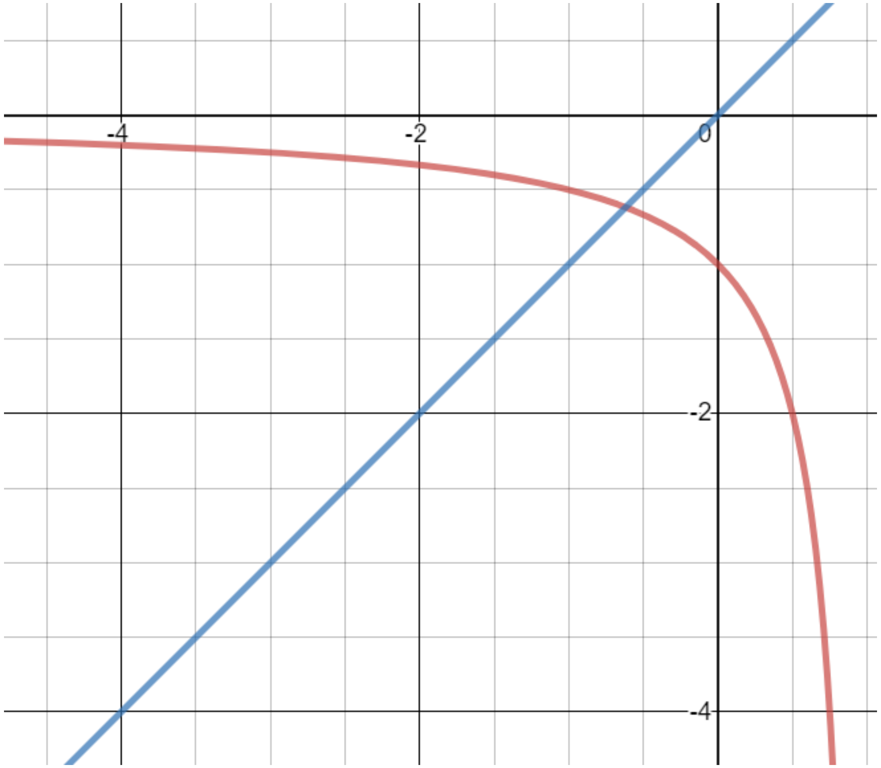
Worked example	Your turn
$f(x) = x^3 + 4x^2 + 3x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$ The equation $f(x) = 0$ has a single root between 1 and 2. (b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places. (c) The root of $f(x) = 0$ is α. By choosing a suitable interval, prove that $\alpha = 1.253$ (3 dp)	$g(x) = x^3 + 3x^2 + 4x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{4(3-x)}{3+x}}, x \neq -3$ The equation $g(x) = 0$ has a single root between 1 and 2. (b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places. (c) The root of $g(x) = 0$ is β. By choosing a suitable interval, prove that $\beta = 1.272$ (3 dp) (c) $g(1.2715) = -0.00821 \ldots < 0$ $g(1.2725) = 0.00827 \ldots > 0$ Sign change and $g(x)$ continuous in the interval $[1.2715, 1.2725]$ $\therefore 1.2715 < \beta < 1.2725$ $\therefore \beta = 1.272$ (3 dp)

Worked example	Your turn
Use the graph of $y = x$ and $y = \sqrt{x+3}$ to solve the equation $$x^2 - x - 3 = 0$$ using the recurrence relation: $$x_{n+1} = \sqrt{x_n + 3}, x_0 = 1$$	Use the graph of $y = x$ and $y = \sqrt{x+1}$ to solve the equation $$x^2 - x - 1 = 0$$ using the recurrence relation: $$x_{n+1} = \sqrt{x_n + 1}, x_0 = 1$$
	
	Staircase diagram converging to root

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Use the graph of $y = x$ and $y = \frac{3}{x-1}$ to solve the equation $$x^2 - x - 3 = 0$$ using the recurrence relation: $$x_{n+1} = \frac{3}{x_n - 1}, x_0 = -4.5$$	Use the graph of $y = x$ and $y = \frac{1}{x-1}$ to solve the equation $$x^2 - x - 1 = 0$$ using the recurrence relation: $$x_{n+1} = \frac{1}{x_n - 1}, x_0 = -2.5$$





Cobweb diagram converging to root

Graphs used with permission from DESMOS: https://www.desmos.com/

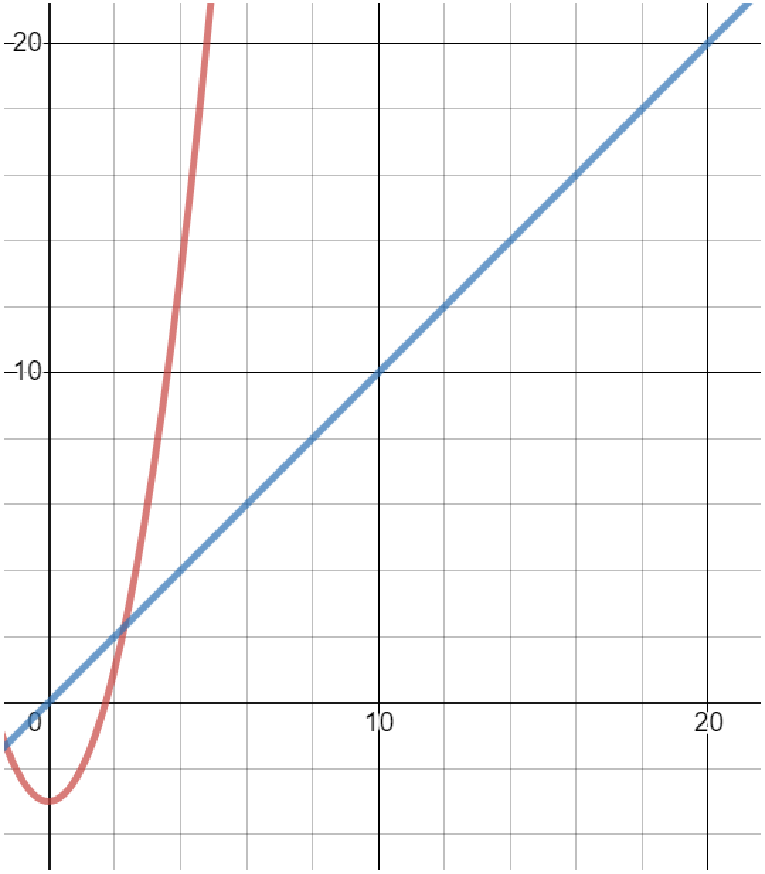
Worked example

Use the graph of $y = x$ and $y = x^2 - 3$ to solve the equation

$$x^2 - x - 3 = 0$$

using the recurrence relation:

$$x_{n+1} = x_n^2 - 3, x_0 = 3$$



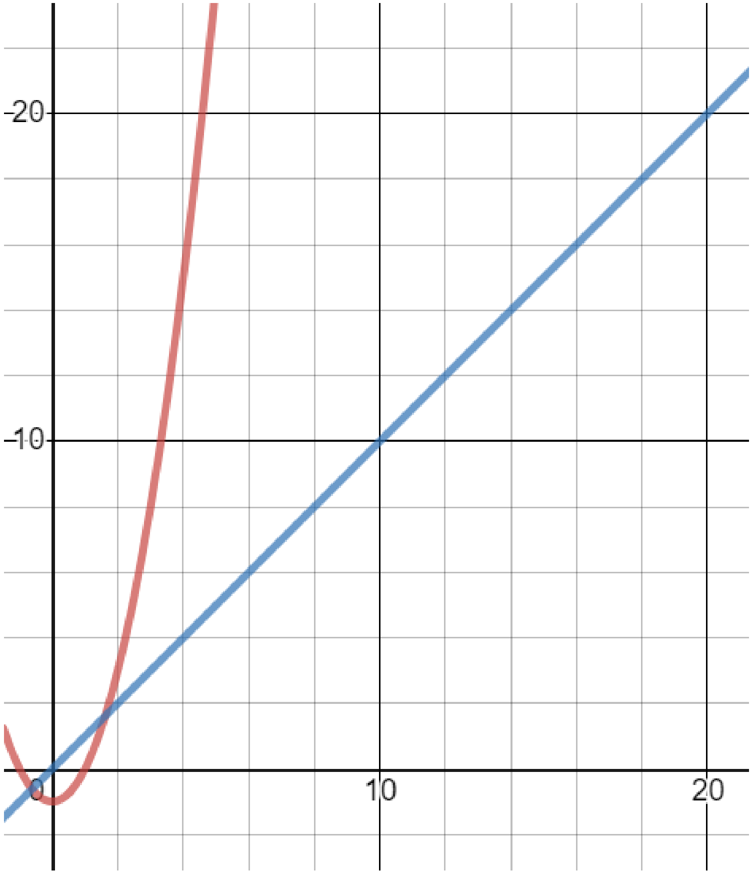
Your turn

Use the graph of $y = x$ and $y = x^2 - 1$ to solve the equation

$$x^2 - x - 1 = 0$$

using the recurrence relation:

$$x_{n+1} = x_n^2 - 1, x_0 = 2$$



Root approximations diverging – iterative method fails

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