10) Numerical methods

10.1) Locating roots
10.2) Iteration
10.3) The Newton-Raphson method
10.4) Applications to modelling

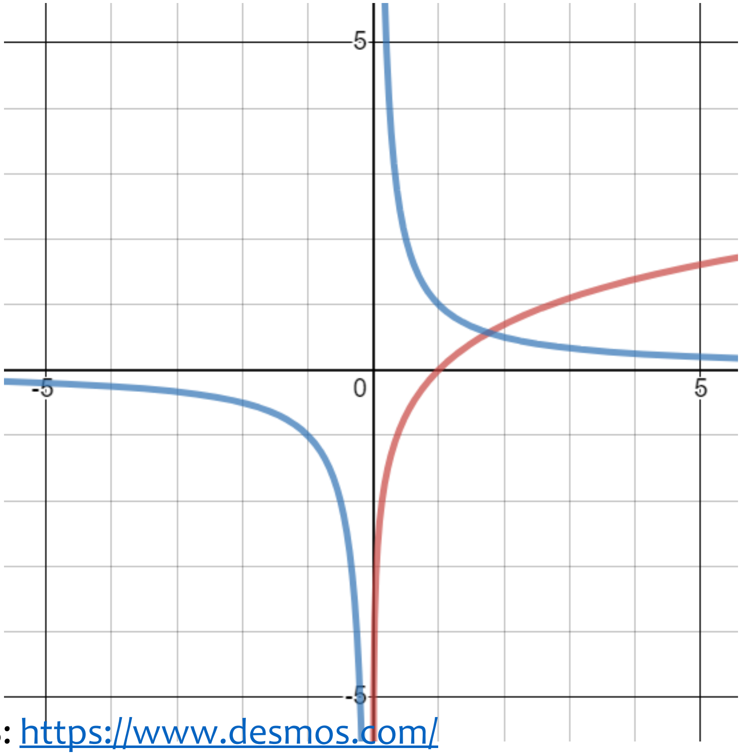
10.1) Locating roots

Chapter CONTENTS

Worked example	Your turn
Show that $f(x) = e^x + 3x - 2$ has a root between $x = 0.2$ and $x = 0.3$	Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$ $f(0.5) = -0.351 \ldots < 0$ $f(0.6) = 0.022 \ldots > 0$ Change of sign and $f(x)$ continuous in the interval $[0.5, 0.6]$ $\therefore$ Root in the interval $[0.5, 0.6]$

Worked example	Your turn
Explain why there are no real roots to $f(x) = \frac{1}{x-2}$ between $x = 1$ and $x = 3$	Explain why there are no real roots to $f(x) = \frac{1}{x}$ between $x = -1$ and $x = 1$ $f(x)$ not continuous in the interval $[-1, 1]$

Worked example	Your turn
Using the same axes, sketch the graphs of $y = e^x$ and $y = \frac{1}{x}$ a) Explain how your diagram shows that the function $f(x) = e^x - \frac{1}{x}$ has only one root	Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$ a) Explain how your diagram shows that the function $f(x) = \ln x - \frac{1}{x}$ has only one root a) The lines intersect where $\ln x = \frac{1}{x} \to \ln x - \frac{1}{x} = 0$ Thus the roots of $f(x)$ correspond to the points of intersection, and there is only one point of intersection on the graph.



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Worked example	Your turn
Using the same axes, sketch the graphs of $y = e^x$ and $y = \frac{1}{x}$ a) Explain how your diagram shows that the function $f(x) = e^x - \frac{1}{x}$ has only one root b) Show that this root lies in the interval $0.5 < x < 0.6$	Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$ a) Explain how your diagram shows that the function $f(x) = \ln x - \frac{1}{x}$ has only one root b) Show that this root lies in the interval $1.7 < x < 1.8$ b) $f(1.7) = -0.0576 \ldots < 0$ $f(1.8) = 0.0322 \ldots > 0$ Change of sign and $f(x)$ continuous in the interval $[1.7, 1.8]$ $\therefore$ Root in the interval $[1.7, 1.8]$

Worked example	Your turn
Using the same axes, sketch the graphs of $y = e^x$ and $y = \frac{1}{x}$ a) Explain how your diagram shows that the function $f(x) = e^x - \frac{1}{x}$ has only one root b) Show that this root lies in the interval $0.5 < x < 0.6$ c) Show that the root is 0.567 to 3 decimal places	Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$ a) Explain how your diagram shows that the function $f(x) = \ln x - \frac{1}{x}$ has only one root b) Show that this root lies in the interval $1.7 < x < 1.8$ c) Show that the root is 1.763 to 3 decimal places c) $f(1.7625) = -0.00064 < 0$ $f(1.7635) = 0.00024 > 0$ Change of sign and $f(x)$ continuous in the interval $[1.7625, 1.7635]$ $\therefore 1.7625 < \alpha < 1.7635,$ $\therefore \alpha = 1.763$ correct to 3dp.

10.2) Iteration

Worked example	Your turn
$f(x) = x^2 - 5x - 3$	$g(x) = x^2 - 6x + 2$
a) Show that $f(x) = 0$ can be written as:	a) Show that $g(x) = 0$ can be written as:
i) $x = \frac{x^2 - 3}{5}$ ii) $x = \sqrt{5x + 3}$ iii) $x = 5 + \frac{3}{x}$	i) $x = \frac{x^2 + 2}{6}$ ii) $x = \sqrt{6x - 2}$ iii) $x = 6 - \frac{2}{x}$
	a) Shown

Worked example

$$f(x) = x^2 - 5x - 3$$

a) Show that $f(x) = 0$ can be written as:

i) $x = \frac{x^2-3}{5}$ ii) $x = \sqrt{5x+3}$ iii) $x = 5 + \frac{3}{x}$

b) Starting with $x_0 = 3$ use each iterative formula to find a root of the equation $f(x) = 0$, rounding your answers to 3 decimal places

Your turn

$$g(x) = x^2 - 6x + 2$$

a) Show that $g(x) = 0$ can be written as:

i) $x = \frac{x^2+2}{6}$ ii) $x = \sqrt{6x-2}$ iii) $x = 6 - \frac{2}{x}$

b) Starting with $x_0 = 4$ use each iterative formula to find a root of the equation $g(x) = 0$, rounding your answers to 3 decimal places

b) i) $x = 0.354$ (3 dp)
ii) $x = 5.646$ (3 dp)
iii) $x = 5.646$ (3 dp)

Worked example	Your turn
$f(x) = e^{x-2} + x - 5$ a) Show that $f(x) = 0$ can be written as: $x = \ln(5 - x) + 2, \qquad x < 5$	$f(x) = e^{x-1} + x - 6$ a) Show that $f(x) = 0$ can be written as: $x = \ln(6 - x) + 1, \qquad x < 6$ a) Shown

Worked example

$$f(x) = e^{x-2} + x - 5$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(5 - x) + 2, \qquad x < 5$$

The root of $f(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(5 - x_n) + 2, \qquad x_0 = 3$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four

decimal places.

Your turn

$$f(x) = e^{x-1} + x - 6$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(6 - x) + 1, \qquad x < 6$$

The root of $f(x) = 0$ is β.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for β

b) Calculate the values of x_1, x_2 and x_3 to four

decimal places.

b)

$x_0 = 2$

$x_1 = \ln(6 - 2) + 1$

$= 2.3863$

$x_2 = 2.2847 \ldots$

$x_3 = 2.3125$

Worked example	Your turn
$f(x) = e^{x-2} + x - 5$	$f(x) = e^{x-1} + x - 6$

Worked example

$$f(x) = e^{x-2} + x - 5$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(5 - x) + 2, \qquad x < 5$$

The root of $f(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(5 - x_n) + 2, \qquad x_0 = 3$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four decimal places.

c) By choosing a suitable interval, show that $\alpha = 2.792$ correct to 3 decimal places.

Your turn

$$f(x) = e^{x-1} + x - 6$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(6 - x) + 1, \qquad x < 6$$

The root of $f(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four decimal places.

c) By choosing a suitable interval, show that $\beta = 2.307$ correct to 3 decimal places.

c)

$f(2.3065) = -0.00027 \ldots < 0$

$f(2.3075) = 0.0044 \ldots > 0$

Sign change and $g(x)$ continuous in the interval $[2.3065, 2.3075]$

$\therefore 2.3065 < \beta < 2.3075$

$\therefore \beta = 2.307$ (3 dp)

Worked example

$$f(x) = x^3 - 5x^2 - 3x + 2$$

(a) Show that the equation $f(x) = 0$ has a root in the interval $5 < x < 6$.

Your turn

$$g(x) = x^3 - 3x^2 - 2x + 5$$

(a) Show that the equation $g(x) = 0$ has a root in the interval $3 < x < 4$.

(a)
$g(3) = -1 < 0$
$g(4) = 13 > 0$
Change of sign and $g(x)$ continuous in the interval $[3, 4]$
$\therefore$ root in the interval $[3, 4]$

Worked example	Your turn
$f(x) = x^3 - 5x^2 - 3x + 2$	$g(x) = x^3 - 3x^2 - 2x + 5$
(a) Show that the equation $f(x) = 0$ has a root in the interval $5 < x < 6$.	(a) Show that the equation $g(x) = 0$ has a root in the interval $3 < x < 4$.
(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 3x_n + 2}{5}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking: (i) $x_0 = 0.5$ (ii) $x_0 = 6$	(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking: (i) $x_0 = 1.5$ (ii) $x_0 = 4$

(b)

i)

$$x_1 = \sqrt{\frac{1.5^3 - 2(1.5) + 5}{3}} = 1.3385\ldots$$

$x_2 = 1.2544\ldots$

$x_3 = 1.2200\ldots$

Convergent as the change in the root on each iteration is decreasing. The iterative method will find the root.

ii)

$$x_1 = \sqrt{\frac{4^3 - 2(4) + 5}{3}} = 4.5092\ldots$$

$x_2 = 5.4058$

$x_3 = 7.1219\ldots$

Divergent as the change in the root on each iteration is increasing. The iterative method has failed to find the root.

Worked example	Your turn
$f(x) = x^3 + 4x^2 + 3x - 12$	$g(x) = x^3 + 3x^2 + 4x - 12$
(a) Show that the equation can be written as	(a) Show that the equation can be written as
$x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$	$x = \sqrt{\frac{4(3-x)}{3+x}}, x \neq -3$
	(a) Shown

Worked example	Your turn
$f(x) = x^3 + 4x^2 + 3x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$ The equation $f(x) = 0$ has a single root between 1 and 2. (b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places.	$g(x) = x^3 + 3x^2 + 4x - 12$ (a) Show that the equation can be written as $x = \sqrt{\frac{4(3-x)}{3+x}}, x \neq -3$ The equation $g(x) = 0$ has a single root between 1 and 2. (b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places. (b) $x_1 = \sqrt{\frac{4(3-1)}{3+1}} = 1.41$ $x_2 = 1.20$ $x_3 = 1.31$

Worked example	Your turn
$f(x) = x^3 + 4x^2 + 3x - 12$	$g(x) = x^3 + 3x^2 + 4x - 12$
(a) Show that the equation can be written as $x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$	(a) Show that the equation can be written as $x = \sqrt{\frac{4(3-x)}{3+x}}, x \neq -3$
The equation $f(x) = 0$ has a single root between 1 and 2.	The equation $g(x) = 0$ has a single root between 1 and 2.
(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places.	(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$ to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places.
(c) The root of $f(x) = 0$ is α. By choosing a suitable interval, prove that $\alpha = 1.253$ (3 dp)	(c) The root of $g(x) = 0$ is β. By choosing a suitable interval, prove that $\beta = 1.272$ (3 dp)
	(c) $g(1.2715) = -0.00821 \ldots < 0$ $g(1.2725) = 0.00827 \ldots > 0$ Sign change and $g(x)$ continuous in the interval $[1.2715, 1.2725]$ $\therefore 1.2715 < \beta < 1.2725$ $\therefore \beta = 1.272$ (3 dp)

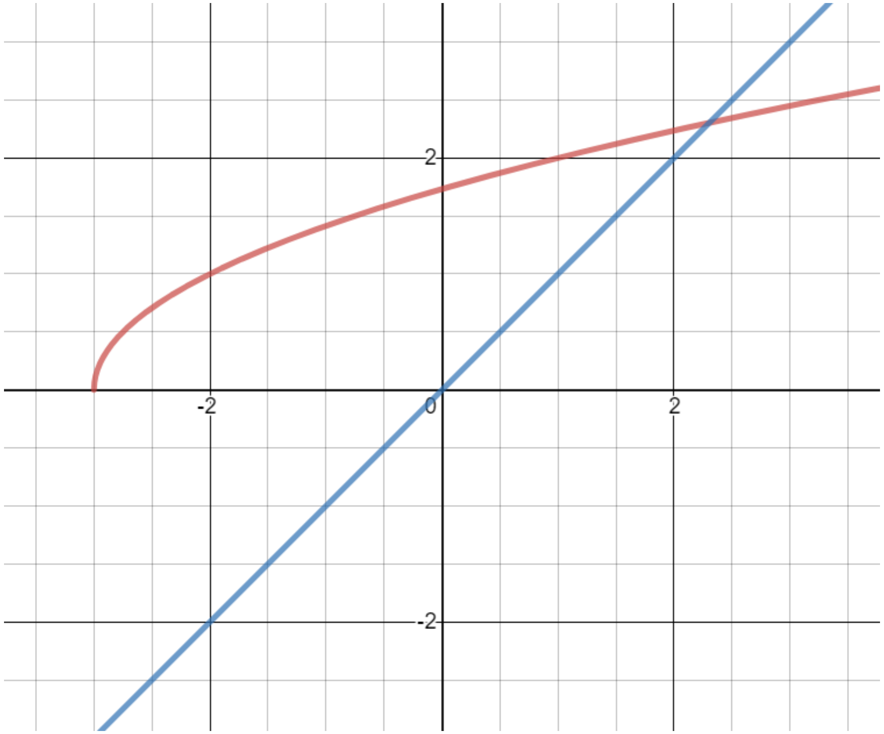
Worked example

Use the graph of $y = x$ and $y = \sqrt{x+3}$ to solve the equation

$$x^2 - x - 3 = 0$$

using the recurrence relation:

$$x_{n+1} = \sqrt{x_n + 3}, x_0 = 1$$



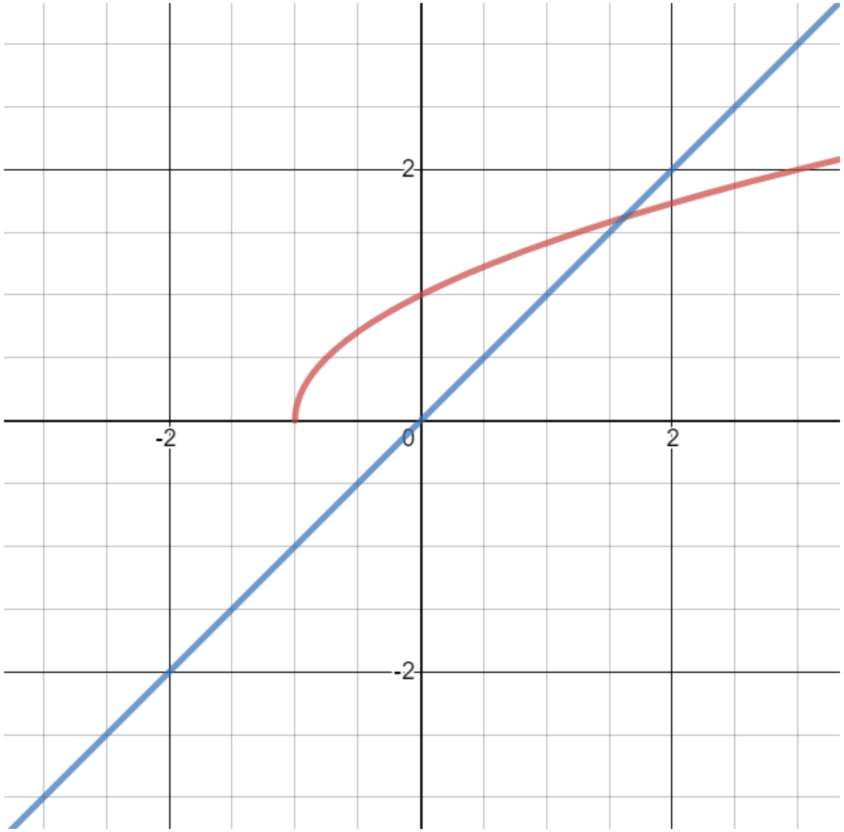
Your turn

Use the graph of $y = x$ and $y = \sqrt{x+1}$ to solve the equation

$$x^2 - x - 1 = 0$$

using the recurrence relation:

$$x_{n+1} = \sqrt{x_n + 1}, x_0 = 1$$



Staircase diagram converging to root

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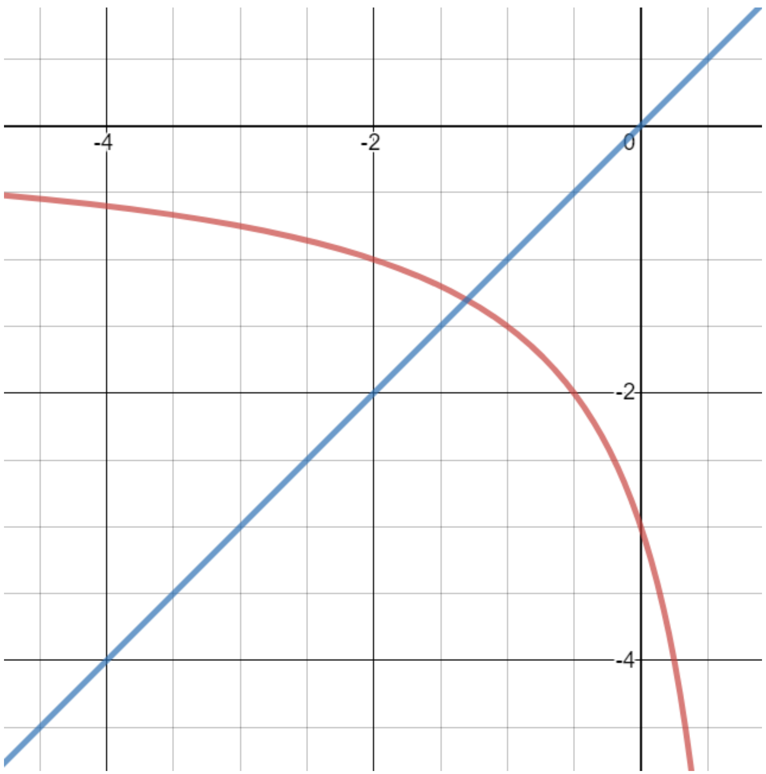
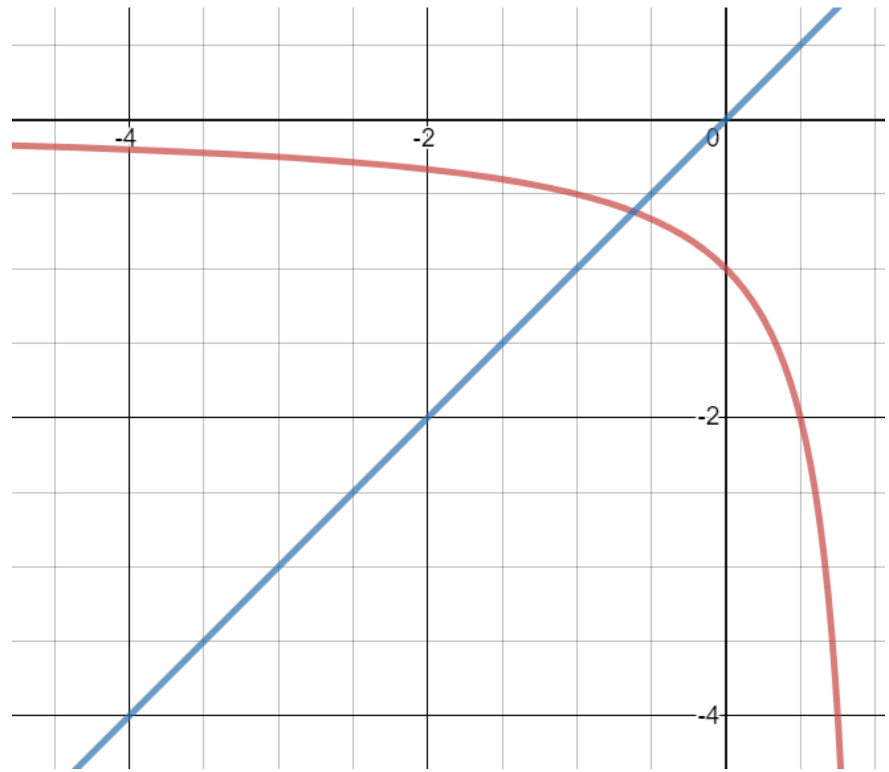
Worked example	Your turn
Use the graph of $y = x$ and $y = \frac{3}{x-1}$ to solve the equation $$x^2 - x - 3 = 0$$ using the recurrence relation: $$x_{n+1} = \frac{3}{x_n - 1}, x_0 = -4.5$$	Use the graph of $y = x$ and $y = \frac{1}{x-1}$ to solve the equation $$x^2 - x - 1 = 0$$ using the recurrence relation: $$x_{n+1} = \frac{1}{x_n - 1}, x_0 = -2.5$$





Cobweb diagram converging to root

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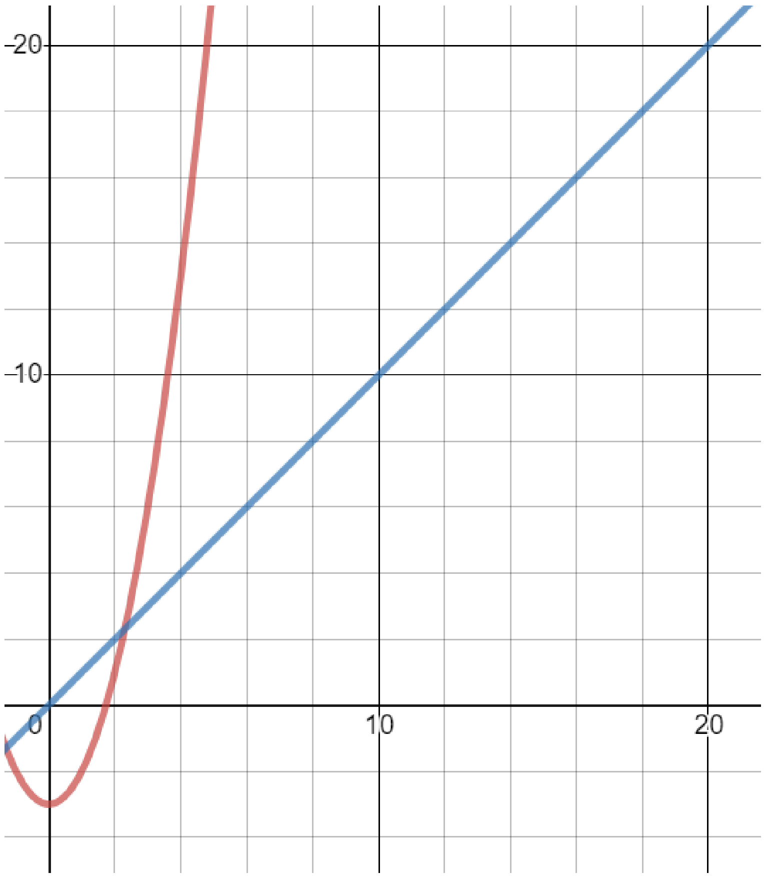
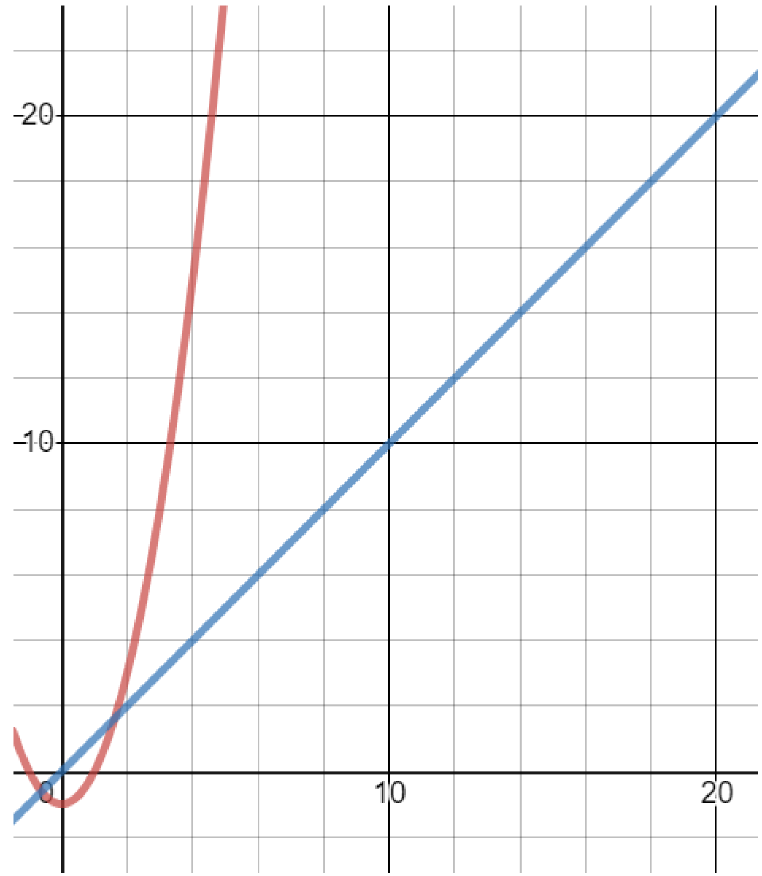
Worked example	Your turn
Use the graph of $y = x$ and $y = x^2 - 3$ to solve the equation $$x^2 - x - 3 = 0$$ using the recurrence relation: $$x_{n+1} = x_n^2 - 3, x_0 = 3$$	Use the graph of $y = x$ and $y = x^2 - 1$ to solve the equation $$x^2 - x - 1 = 0$$ using the recurrence relation: $$x_{n+1} = x_n^2 - 1, x_0 = 2$$





Root approximations diverging – iterative method fails

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10.3) The Newton-Raphson method

Worked example	Your turn
Using the Newton-Raphson process, state the recurrence relation for the following functions:	Using the Newton-Raphson process, state the recurrence relation for the following functions:
$f(x) = x^4 - 3$	$f(x) = x^3 - 2$
	$x_{n+1} = x_n - \frac{(x_n)^3 - 2}{3(x_n)^2}$
$g(x) = \sec x$	$g(x) = \tan x$
	$x_{n+1} = x_n - \frac{\tan x_n}{\sec^2 x_n} = x_n - \frac{1}{2}\sin(2x_n)$
$h(x) = x^2 + x + 3$	$h(x) = x^2 - x - 1$
	$x_{n+1} = x_n - \frac{(x_n)^2 - x_n - 1}{2x_n - 1}$

Worked example	Your turn
Using three iterations of the Newton-Raphson process, starting with $x_0 = 0.5$, solve the equation $$x = \sin x$$	Using three iterations of the Newton-Raphson process, starting with $x_0 = 0.5$, solve the equation $$x = \cos x$$ Let $f(x) = x - \cos x$ $f'(x) = 1 + \sin x$ $x_{n+1} = x_n - \frac{x_n - cos(x_n)}{1 + sin(x_n)}, x_0 = 0.5$ $x_1 = 0.5 - \frac{0.5 - cos(0.5)}{1 + sin(0.5)} = 0.7552224 \ldots$ $x_2 = 0.7391412$ $x_3 = 0.7390851$ $x = 0.739$ (3 dp)

Worked example	Your turn
$$f(x) = \frac{1}{3}x^4 - x^2 + 3x - 1$$ The equation $f(x) = 0$ has a root α in the interval $[-2, -3]$ Taking -2.5 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 2 decimal places.	$$g(x) = \frac{1}{2}x^4 - x^3 + x - 3$$ The equation $g(x) = 0$ has a root β in the interval $[-2, -1]$ Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $g(x)$ to obtain a second approximation to β. Give your answer to 2 decimal places. $$x_{n+1} = x_n - \frac{\frac{1}{2}(x_n)^4 - (x_n)^3 + x_n - 3}{2(x_n)^3 - 3(x_n)^2 + 1}$$ $$\beta_1 = -1.5 - \frac{1.40625}{-12.5} =- -1.3875 \ldots$$ $= 1.39$ (2 dp)

Worked example	Your turn
$$f(x) = 11x^2 - \frac{3}{x^2}$$ The equation $f(x) = 0$ has a root α in the interval $[0, 1]$ Taking 0.4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.	$$g(x) = 3x^2 - \frac{11}{x^2}$$ The equation $g(x) = 0$ has a root β in the interval $[1, 2]$ Taking 1.4 as a first approximation to β, apply the Newton-Raphson process once to $g(x)$ to obtain a second approximation to β. Give your answer to 3 decimal places.

$$x_{n+1} = x_n - \frac{3(x_n)^2 - \frac{11}{(x_n)^2}}{6x_n + \frac{22}{(x_n)^3}}$$

$$\beta_1 = 1.4 - \frac{0.2677\ldots}{16.4174\ldots} = 1.38369\ldots$$
$$= 1.384 \text{ (3 dp)}$$

Worked example	Your turn
$f(x) = x^2 - 5x + 8$ State why $x_0 = 2.5$ is not suitable to use as a first approximation to the roots of $f(x)$ when applying the Newton-Raphson method.	$f(x) = x^2 + 7x + 8$ State why $x_0 = -3.5$ is not suitable to use as a first approximation to the roots of $f(x)$ when applying the Newton-Raphson method. $f'(x) = 2x + 7 = 0$ -> $x = -3.5$ Turning point at $x = -3.5$ $f'(-3.5) = 0$ You cannot divide by 0 in the Newton-Raphson method. Also the tangent to $y = f(x)$ at $x = -3.5$ would be horizontal, and therefore never intersect the x-axis.

10.4) Applications to modelling

Worked example

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 5000\,(0.58)^x - 100\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 5 years after purchase.

Your turn

The price of a car in £s, x years after purchase, is modelled by the function

$$g(x) = 15000\,(0.85)^x - 1000\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(a) £3500

Worked example	Your turn
The price of a car in £s, x years after purchase, is modelled by the function $f(x) = 5000\,(0.58)^x - 100\sin x\,, \qquad x > 0$ (a) Use the model to find the value, to the nearest hundred £s, of the car 5 years after purchase. (b) Show that $f(x)$ has a root between 7 and 8.	The price of a car in £s, x years after purchase, is modelled by the function $g(x) = 15000\,(0.85)^x - 1000\sin x\,, \qquad x > 0$ (a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase. (b) Show that $g(x)$ has a root between 19 and 20. (b) $g(19) = 543.11 \ldots > 0$ $g(20) = -331.55 \ldots < 0$ Change of sign and $g(x)$ continuous in the interval $[19, 20]$ $\therefore$ root in the interval $[19, 20]$

Worked example	Your turn
The price of a car in £s, x years after purchase, is modelled by the function	The price of a car in £s, x years after purchase, is modelled by the function

Worked example

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 5000\,(0.58)^x - 100\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 5 years after purchase.

(b) Show that $f(x)$ has a root between 7 and 8.

(c) Taking 7.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

Your turn

The price of a car in £s, x years after purchase, is modelled by the function

$$g(x) = 15000\,(0.85)^x - 1000\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $g(x)$ has a root between 19 and 20.

(c) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $g(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(c)

$g'(x) = (15\,000)(0.85)^x(\ln 0.85) - 1000\cos x$

$g(19.5) = 15000(0.85)^{19.5} - 1000\sin(19.5)$

$= 25.0693\ldots$

$g'(19.5) = 15000(0.85)^{19.5}(\ln 0.85) - 1000\cos(19.5)$

$= -893.3009\ldots$

$x_1 = 19.5 - \frac{g(19.5)}{g'(19.5)} = 19.528$ (3 dp)

Worked example

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 5000\,(0.58)^x - 100\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 5 years after purchase.

(b) Show that $f(x)$ has a root between 7 and 8.

(c) Taking 7.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(d) Criticise this model with respect to the value of the car as it gets older.

Your turn

The price of a car in £s, x years after purchase, is modelled by the function

$$g(x) = 15000\,(0.85)^x - 1000\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $g(x)$ has a root between 19 and 20.

(c) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $g(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(d) Criticise this model with respect to the value of the car as it gets older.

(d) In reality, the car can never have a negative value so this model is not reasonable for cars that are approximately 20 or more years old.