Elastic Strings and Springs (Chapter 3)

Hooke's Law: When an elastic string or spring is stretched, the tension, T, produced is proportional to the extension, x.

or compression

$$T = \frac{\lambda}{l}x$$

- λ is known as 'the modulus of elasticity' (see note)
- l is the 'natural (unstretched) length' of the string/spring.

Note about λ:

- In A level Maths 'the modulus of elasticity' refers to the constant λ in the equation $T=\frac{\lambda}{l}x$.
- λ can be understood as the force needed to double the length of a string/spring: hence, the units of λ are Newtons (N)
- λ should not be confused with 'Young's modulus' – a constant that does not feature in A level maths, but which you might have come across in physics (and which is confusingly sometimes referred to as the 'elastic modulus').

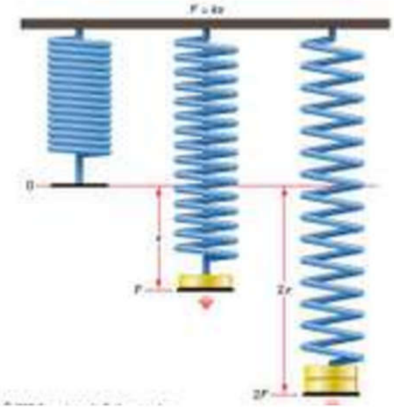
Hooke's Law: When it Applies

Many springs, strings, wires and solids obey Hooke's Law in a limited range. Hooke's law only ever applies up to a maximum force known as the *elastic limit* which varies for each spring etc. Under very large forces springs deform and break.

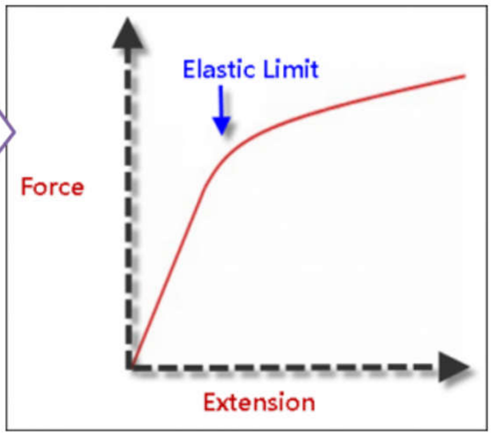
If a string or spring obeys Hooke's Law it is called *elastic*.

An elastic spring can also be compressed (producing thrust instead of tension). However, elastic strings don't resist compression (think about how string behaves).

All string/springs (in A Level Maths) are light - they don't extend under their own weight.



Graph showing the extension of a Spring as the weight hung from it, and hence the tension in the spring, increases.
The variables are in direct proportion (obeying Hooke's Law) up to the 'elastic limit' beyond which the law doesn't hold.



An elastic string of natural length 2m and modulus of elasticity 29.4N has one end fixed. A particle of mass 4kg is attached to the other end and hangs at rest. Find the extension of the string.

An elastic spring of natural length 1.5m has one end attached to a fixed point. A horizontal force of magnitude 6N is applied to the other end and compresses the spring to a length of 1m.

Find the modulus of elasticity of the spring.

A string of natural length l and modulus of elasticity 123N is stretched to a length $2l$. What is the Tension in the string?

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A string of natural length l and modulus of elasticity 123N is extended by a distance $2l$. What is the Tension in the string?

A spring of natural length 3m is stretched to a length of 6m by applying a force to one end of 99N. What is the modulus of elasticity of the spring?

A spring of natural length l and $\lambda = 40N$ is compressed to a length $\frac{3}{4}l$. What is the compressive force?

A string of natural length l and $\lambda = 40N$ is compressed by a force 30N. What is x?

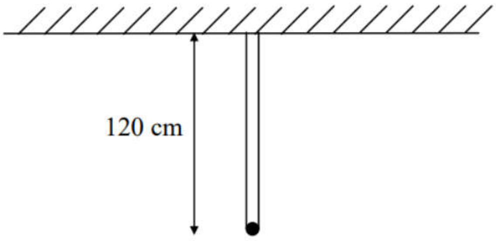
Combining strings/springs

The elastic springs PQ and QR are joined together at Q to form one long spring. The spring PQ has natural length 1.6m and modulus of elasticity 20N. The spring QR has natural length 1.4m and modulus of elasticity 28N. The ends P and R, of the long string are attached to two fixed points which are 4m apart on the same horizontal plane. [*Assume Q is at rest and in equilibrium*].
Find the tension in the combined spring.

Two identical elastic springs PQ and QR each have natural length l and modulus of elasticity $2mg$. The springs are joined together at Q. Their other ends, P and R, are attached to fixed points, with P being $4l$ vertically above R.
A particle of mass m is attached at Q and hangs at rest in equilibrium.
Find the distance of the particle below P.

Your Turn

Figure 1



A particle of mass 5 kg is attached to one end of two light elastic strings. The other ends of the strings are attached to a hook on a beam. The particle hangs in equilibrium at a distance 120 cm below the hook with both strings vertical, as shown in Fig. 1. One string has natural length 100 cm and modulus of elasticity 175 N. The other string has natural length 90 cm and modulus of elasticity λ newtons.

Find the value of λ.

(5)

Ex 3A Q1-7
Skip 5

Slopes, Friction, Moments, Resolving Forces, etc.

Elastic Strings and Springs can be introduced into similar problems to those encountered in A level Mechanics. e.g. slopes, friction, **moments (not yet studied)**

An elastic string has natural length 2m and modulus of elasticity 98N.
One end of the string is attached to a fixed point *O* and the other end is attached to a particle P of mass 4kg.
The particle is held in equilibrium by a horizontal force of magnitude 28N, with *OP* making an angle α with the vertical.
Find:
a) the value of α
b) The length *OP*

An elastic string of natural length $2l$ and modulus of elasticity $4mg$ is stretched between two points A and B.
The points A and B are on the same horizontal level and AB $=2l$.
A particle P is attached to the midpoint of the string and hangs in equilibrium with both parts of the string making an angle of 30° with the line AB.
Find, in terms of m, the mass of the particle.

we'll need a different m... use M instead

A particle P, of mass m, rests in equilibrium on a rough plane inclined at 30° to the horizontal. The coefficient of friction between the particle and the plane is $\frac{\sqrt{3}}{3}$. P is attached to a fixed point A on the plane by a light elastic spring with natural length a and modulus of elasticity $3mg$. P is free to move only in a straight line below A down the line of greatest slope. Write an inequality for the length AP.

One end, *A*, of a light elastic string *AB*, of natural length 0.6m and modulus of elasticity 10N, is fixed to a point on a fixed rough plane inclined at an angle θ to the horizontal, where $\sin\theta = 0.8$. A ball of mass 3kg is attached to the end, *B*, of the string. The coefficient of friction, μ, between the ball and the plane is 1/3. The ball rests in limiting equilibrium, on the point of sliding down the plane, with *AB* along the line of greatest slope.

Find:

a) The tension in the string

b) The length of the string

c) If $\mu > 1/3$, without doing any further calculation, state how your answer to b) would change

Ex 3A

9, 10, 11

Hooke's Law and Dynamic Problems

Elastic Strings and Springs can be introduced into more similar problems to those encountered in A level Mechanics.

Key ideas to Remember:

- **F**=m**a**
- Maximum displacement occurs when velocity = 0 } (in a given direction)
- Maximum velocity occurs when acceleration = 0 } (in a given direction)
- $Fr_{max} = \mu R$

One end of a light elastic string, of natural length 0.5m and modulus of elasticity 20N, is attached to a fixed point A. The other end is attached to a particle of mass 2kg. The particle is held at a point which is 1.5m below A and released from rest. Find:

a) The initial acceleration of the particle

b) The length of the string when the particle reaches its maximum speed.

A particle of mass 0.5kg is attached to one end of a light elastic spring of natural length 1.5m and modulus of elasticity 19.6N. The other end of the spring is attached to a fixed point *O* on a rough plane which is inclined to the horizontal at an angle α where tan α = 3/4.
The coefficient of friction between the particle and the plane is 0.2. The particle is held at rest on the plane at a point which is 1m from *O* down a line of greatest slope of the plane.
The particle is released from rest and moves down the slope. Find its initial acceleration.

Ex 3B
1-2, 4-5

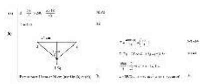
Edexcel M3 Jan 2013 Q7

A particle P of mass 1.5 kg is attached to the mid-point of a light elastic string of natural length 0.30 m and modulus of elasticity λ newtons. The ends of the string are attached to two fixed points A and B, where AB is horizontal and AB = 0.48 m. Initially P is held at rest at the mid-point, M, of the line AB and the tension in the string is 240 N.

(*a*) Show that $\lambda = 400$. **(3)**

The particle is now held at rest at the point C, where C is 0.07 m vertically below M. The particle is released from rest at C.

(*b*) Find the magnitude of the initial acceleration of P. **(6)**



Ex 3B
3

Elastic Potential Energy

✎E.P.E. $= \frac{\lambda x^2}{2l}$

The Elastic Potential Energy (E.P.E.) stored in a spring or string is equal to the work done in stretching/compressing it.

Work done =

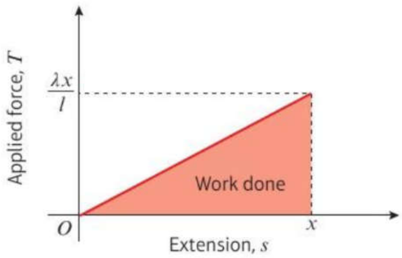
Initial tension (before extending) =

Final tension (after extending) =

Average tension =

EPE = average tension x distance =

Area of triangle = work done



Integration:

$$E.P.E. = \int_0^x T\,ds = \int_0^x \frac{\lambda s}{l}\,ds = \left[\frac{\lambda s^2}{2l}\right]_0^x = \frac{\lambda x^2}{2l}$$

The variable s represents the extension at any point during stretching. x represents the final extension.

An elastic string has natural length 1.4m and modulus of elasticity 6N. Find the energy stored in the string when its length is 1.6m

A light elastic spring has natural length 0.6m and modulus of elasticity 10N. Find the work done in compressing the spring from a length of 0.5m to a length of 0.3m.

Ex 3C - quick!

The Law of Conservation of Energy (expanded)

Initial Energy = Final Energy

work done by engine +

initial EPE + initial GPE + initial KE **= final EPE** + final GPE + final KE + w.d. against friction

Potential energy released/energy going in	=	Potential energy stored/energy going out

Consider the energy it has at the beginning - I tend to think of this as the energy it has in the bank (a bit like money).

Some of this energy is 'spent' in various ways - it is either spent and converted into another type of energy - or it is spent on having to overcome friction/resistance. Some energy is not spent, but is instead increased (e.g. the KE may increase if it gets faster because GPE is converted to KE)

If there is a force/engine doing work, then there is more energy 'in the bank' to be converted. This is why it is on the LHS of the equation.

A light elastic string, of natural length 1.6m and modulus of elasticity 10N, has one end fixed at a point A on a smooth horizontal table. A particle of mass 2kg is attached to the other end of the string. The particle is held at the point A and projected horizontally along the table with speed $2ms^{-1}$. Find how far it travels before first coming to instantaneous rest.

A particle of mass 0.5kg is attached to one end of an elastic string, of natural length 2m and modulus of elasticity 19.6N. The other end of the elastic string is attached to a point O. The particle is released from the point O. Find the greatest distance it will reach below O.

A light elastic spring, of natural length 1m and modulus of elasticity 10N, has one end attached to a fixed point *A*. A particle of mass 2kg is attached to the other end of the spring and is held at a point *B* which is 0.8m vertically below *A*. The particle is projected vertically downwards from *B* with speed $2ms^{-1}$. Find the distance it travels before first coming to rest.

A light elastic spring, of natural length 0.5m and modulus of elasticity 10N, has one end attached to a point *A* on a rough horizontal plane. The other end is attached to a particle *P* of mass 0.8kg. The coefficient of friction between the particle and the plane is 0.4. The particle initially lies on the plane with *AP* = 0.5m and is then projected with speed $2ms^{-1}$ away from *A* along the plane. Find the distance moved by P before it first comes to rest.

Edexcel M3 June 2014(R) Qu3

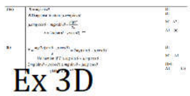
One end A of a light elastic string AB, of modulus of elasticity mg and natural length a, is fixed to a point on a rough plane inclined at an angle θ to the horizontal. The other end B of the string is attached to a particle of mass m which is held at rest on the plane. The string AB lies along a line of greatest slope of the plane, with B lower than A and $AB = a$. The coefficient of friction between the particle and the plane is μ, where $\mu < \tan\theta$. The particle is released from rest.

(*a*) Show that when the particle comes to rest it has moved a distance $2a(\sin\theta - \mu\cos\theta)$ down the plane.

(6)

(*b*) Given that there is no further motion, show that $\mu \geq \frac{1}{3}\tan\theta$.

(5)



Ex 3D

9 Two points A and B are on the same horizontal level with $AB = 3a$. A particle P of mass m is joined to A by a light inextensible string of length $4a$ and is joined to B by a light elastic string, of natural length a and modulus of elasticity $\frac{mg}{4}$. The particle P is held at a point C, such that $BC = a$ and both strings are taut. The particle P is released from rest.

a Show that when AP is vertical the speed of P is $2\sqrt{ga}$. **(6 marks)**

b Find the tension in the elastic string in this position. **(4 marks)**