CP2 Chapter 6

Hyperbolic Functions

Course Structure

1. Definition of hyperbolic functions and their sketches.
2. Inverse hyperbolic functions.
3. Hyperbolic Identities and Solving Equations
4. Differentiation
5. Integration

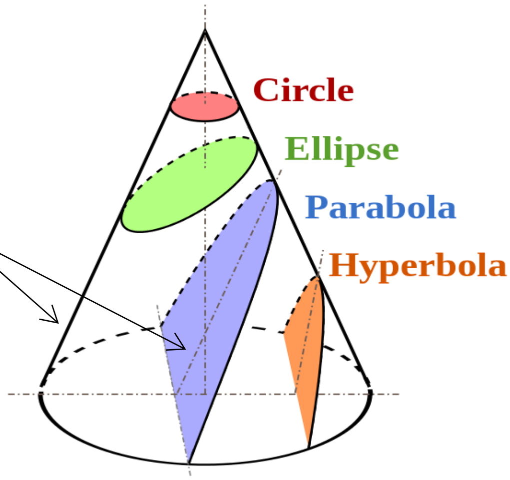
8 **Hyperbolic functions**	8.1	Understand the definitions of hyperbolic functions sinh x, cosh x and tanh x, including their domains and ranges, and be able to sketch their graphs.	For example, $\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$
	8.2	Differentiate and integrate hyperbolic functions.	For example, differentiate $\tanh 3x$, $x \sinh^2 x$, $\frac{\cosh 2x}{\sqrt{(x+1)}}$
8 **Hyperbolic functions** *continued*	8.3	Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.	$\text{arsinh}x = \ln\left[x + \sqrt{x^2+1}\right]$ $\text{arcosh}x = \ln\left[x + \sqrt{x^2-1}\right], \ x \geqslant 1$ $\text{artanh}x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right], \ -1 < x < 1$
	8.4	Derive and use the logarithmic forms of the inverse hyperbolic functions.	
	8.5	Integrate functions of the form $\left(x^2+a^2\right)^{-\frac{1}{2}}$ and $\left(x^2-a^2\right)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.	

<u>Conic Sections</u>

In mathematics there are a number of different **families of curves**. Each of these have different properties and their equations have different forms.

It is possible to obtain these different types of curves by **slicing a cone**, hence "conic sections".

The axis of the parabola is parallel to the side of the cone.

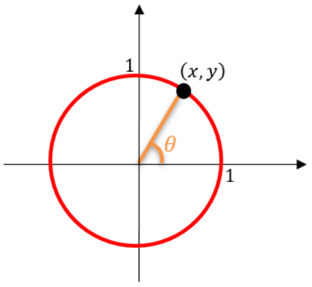


For interest:

Comparing circles and hyperbolas

(Don't make notes on this slide) You will cover Hyperbolas in FP1, but this will give some context for the eponymously named 'hyperbolic functions' that we will explore in this chapter.

Circles



The 'simplest' circle is a unit circle centred at the origin.

Cartesian equation:

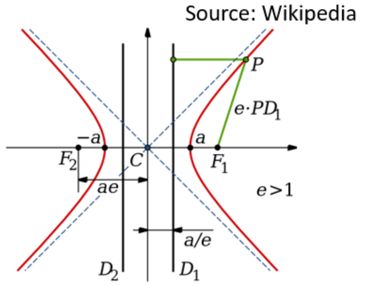
$$\boldsymbol{x^2 + y^2 = 1}$$

Parametric eqns (in terms of θ):

$$\boldsymbol{x = \cos\theta}$$
$$\boldsymbol{y = \sin\theta}$$

Hyperbolas

Source: Wikipedia



The equivalent hyperbola (which crosses x-axis at $(1, 0)$ and $(-1, 0)$)

Cartesian equation:

$$\boldsymbol{x^2 - y^2 = 1}$$

Parametric equations:

$$\boldsymbol{x = \cosh\theta}$$
$$\boldsymbol{y = \sinh\theta}$$

What's the point of hyperbolic functions?

Hyperbolic functions often result from differential equations (e.g. in mechanics), and we'll see later in this module how we can use these functions in calculus.

For example, we can consider forces acting on each point on a hanging piece of string.

Solving the relevant differential equations, we end up with coshx.

Equations for Hyperbolic Functions

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad x \in \mathbb{R}$$

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad x \in \mathbb{R}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad x \in \mathbb{R}$$

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad x \in \mathbb{R}$$

Hyperbolic cosecant:

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad x \in \mathbb{R},\ x \neq 0$$

Hyperbolic cotangent:

$$\coth x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \qquad x \in \mathbb{R},\ x \neq 0$$

Examples:

1. Calculate (using both your $sinh$ button and using the formula)

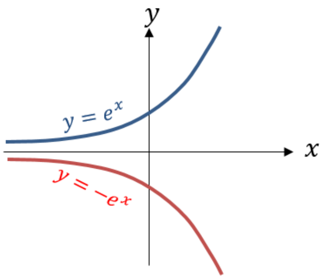
$$\mathbf{\sinh 3 = 10.02}$$

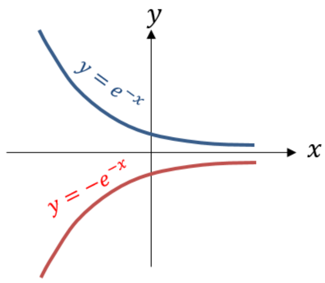
2. Write in terms of e: $\mathbf{cosech\ 3}$

3. Find the exact value of: $\boldsymbol{tanh\ (ln\ 4)}$

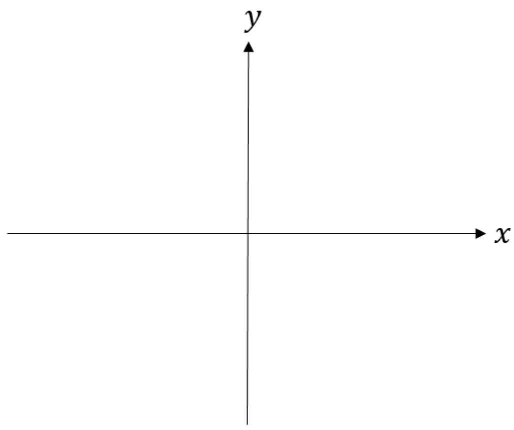
4. Solve $sinh\ x = 5$

<u>Sketching Hyperbolic Functions</u>

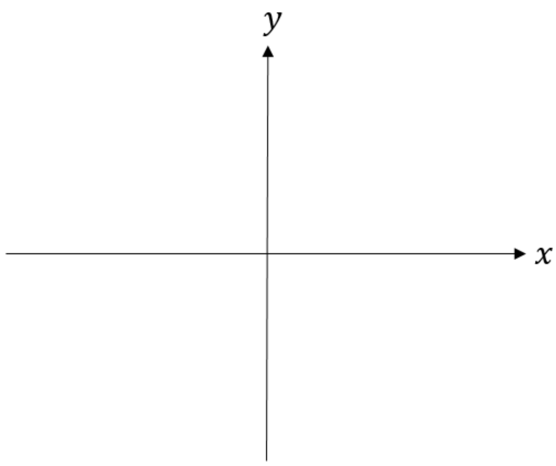




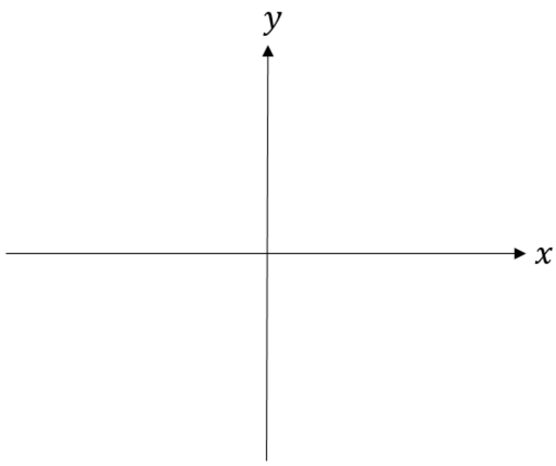
- $y = \sinh x$ is the average of e^x and $-e^{-x}$:



- $y = \cosh x$ is the average of e^x and e^{-x}:

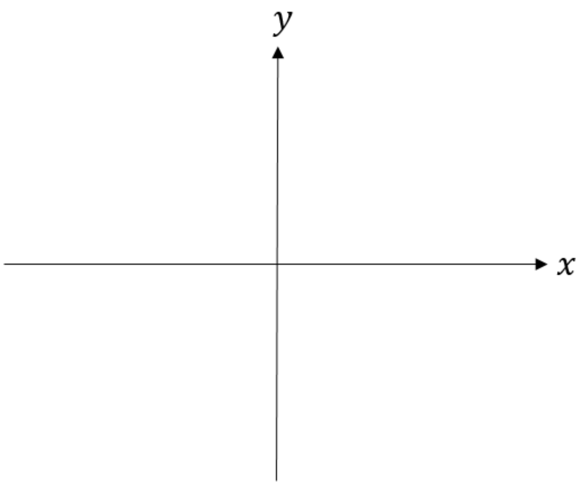


- $\tanh x = \dfrac{\sinh x}{\cosh x}$



Test Your Understanding

Sketch the graph of $y = \operatorname{sech} x$



[FP3 June 2011 Q5] The curve C_1 has equation $y = 3 \sinh 2x$, and the curve C_2 has equation $y = 13 - 3e^{2x}$.

(*a*) Sketch the graph of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes.

(4)

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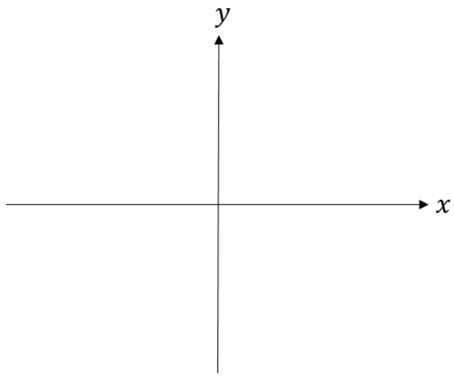
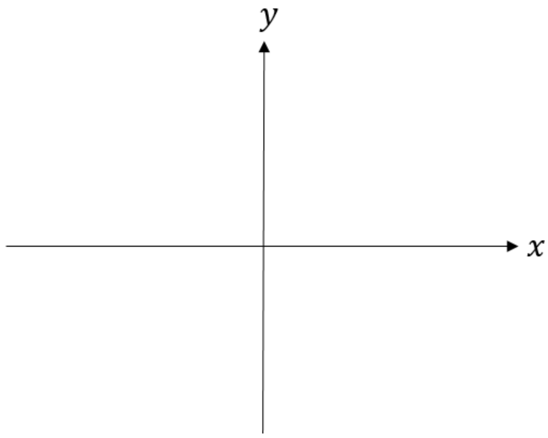
Inverse Hyperbolic Functions

Each hyperbolic function has an inverse.

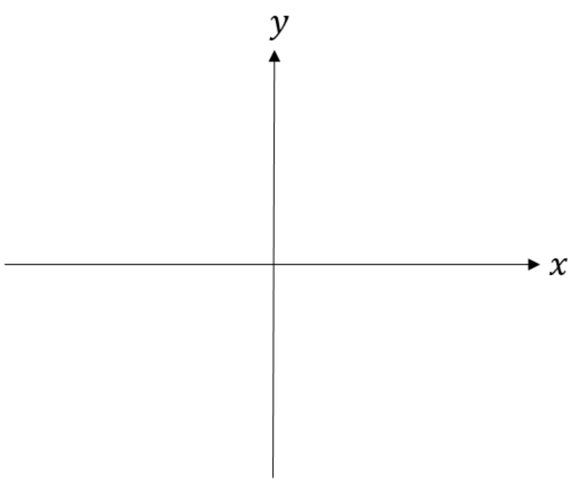
$\boldsymbol{y = arsinh\ x}$,



$\boldsymbol{y = arcosh\ x}$,



$\boldsymbol{y = artanh\ x}$



$\boldsymbol{y = arsech\ x}$,

$\boldsymbol{y = arcosech\ x}$,

$\boldsymbol{y = arcoth\ x}$

Expressing Inverse Hyperbolic Functions in terms of ln

Given that hyperbolic functions can be written in terms of e inverse hyperbolic can be expressed in terms of ln.

Example:

Prove that $arsinh\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Test Your Understanding

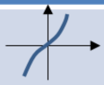
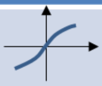
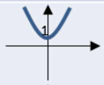
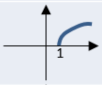
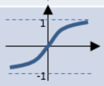
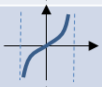
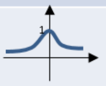
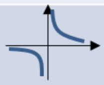
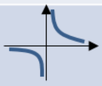
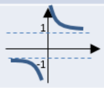
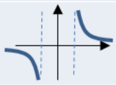
Prove that $arcosh\, x = \ln\left(x + \sqrt{x^2 - 1}\right),\ x \geq 1$

Summary so far:

$$arsinh\, x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$arcosh\, x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$$

$$artanh\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$$

Hyperbolic	Domain	Sketch	Inverse Hyperbolic	Domain	Sketch
$y = \sinh x$	$x \in \mathbb{R}$		$y = arsinh\, x$	$x \in \mathbb{R}$	
$y = \cosh x$	$x \geq 0$		$y = arcosh\, x$	$x \geq 1$	
$y = \tanh x$	$x \in \mathbb{R}$		$y = artanh\, x$	$\|x\| < 1$	
$y = \mathrm{sech}\, x$	$x \geq 0$		$y = arsech\, x$	$0 < x \leq 1$	
$y = cosech\, x$	$x \neq 0$		$y = arcosech\, x$	$x \neq 0$	
$y = \coth x$	$x \neq 0$		$y = \mathrm{arcoth}\, x$	$\|x\| > 1$	

Ex 6B pg 125

Hyperbolic Identities

Use the definitions of $sinh$ and $cosh$ to find:

$\mathbf{cosh^2\, x - sinh^2\, x =}$

$\mathbf{1 - tanh^2 x =}$

$\mathbf{coth^2 x - 1 =}$

Also:

$$\mathbf{sinh\,(A \pm B) = sinh\,A\,cosh\,B \pm cosh\,A\,sinh\,B}$$

$$\mathbf{cosh\,(A \pm B) = cosh\,A\,cosh\,B \pm sinh\,A\,sinh\,B}$$

Hence:

$\mathbf{tanh\,(A \pm B) =}$

Osborn's Rule

We can get these identities from the normal sin/cos ones using Osborn's rule:

Osborn's Rule:

1. Replace $sin \rightarrow sinh$ and $cos \rightarrow cosh$
2. **Negate** any explicit or implied **product of two sines**.

$$\sin A \sin B =$$

$$\tan^2 A \quad =$$

$$cos2A = 2\cos^2 A - 1 \quad \rightarrow$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \rightarrow$$

Solving Equations

To solve equations either use hyperbolic identities or basic definitions of hyperbolic functions.

Examples

1. Solve for all real x, $6 \sinh x - 2 \cosh x = 7$

2. Solve for all real x, $2\cosh^2 x - 5sinh\, x = 5$

Recap: If $\cos x = \frac{3}{5}$, find $\sin x$

3. If $\sinh x = \frac{3}{4}$, find the exact value of:
   a) $\cosh x$
   b) $\tanh x$
   c) $\sinh 2x$

Test Your Understanding

1.

[FP3 June 2009 Q1] Solve the equation

$$7 \operatorname{sech} x - \tanh x = 5$$

Give your answers in the form $\ln a$, where a is a rational number. **(5)**

2.

[FP3 June 2014 (I) Q3] Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials,

(*a*) prove that

$$\cosh^2 x - \sinh^2 x \equiv 1$$

(2)

(*b*) find algebraically the exact solutions of the equation

$$2 \sinh x + 7 \cosh x = 9$$

giving your answers as natural logarithms. **(5)**

3.

[FP3 June 2011 Q5]

(*b*) Solve the equation $3 \sinh 2x = 13 - 3e^{2x}$, giving your answer in the form $\frac{1}{2} \ln k$, where k is an integer.

(5)

Ex 6C, pg 128

Differentiating hyperbolic functions

$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(\tanh x) = \text{sech}^2\, x$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(\coth x) = -cosech^2 x$

Example

Prove that $\frac{d}{dx}(\sinh x) = \cosh x$

Test Your Understanding

[June 2014 (R) Q3] 6. The curve C has equation

$$y = \frac{1}{2}\ln(\coth x), \qquad x > 0$$

(*a*) Show that

$$\frac{dy}{dx} = -\text{cosech}\, 2x$$ **(3)**

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$

Proof of $\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$

Examples

1. Find $\frac{d}{dx}(artanh\ 3x)$

2. Given that $y = (arcosh\, x)^2$ prove that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y$

Test Your Understanding

[June 2009 Q4] Given that $y = \text{arsinh}(\sqrt{x}),\quad x > 0,$

(*a*) find $\frac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a simplified fraction. **(3)**

[June 2010 Q5] Given that $y = (\text{arcosh } 3x)^2$, where $3x > 1$, show that

(*a*) $(9x^2 - 1)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 36y$, **(5)**

(*b*) $(9x^2 - 1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9x\frac{\mathrm{d}y}{\mathrm{d}x} = 18$. **(4)**

Using Maclaurin expansions for approximations

Textbook Example

(a) Show that $\frac{d}{dx}(arsinh\ x) = \frac{1}{\sqrt{1+x^2}}$ **[We did this earlier]**

(b) Find the first two non-zero terms of the series expansion of $arsinh\ x$.

The general form for the series expansion of $arsinh\ x$ is given by

$$arsinh\ x = \sum_{r=0}^{\infty} \left(\frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \right) \frac{x^{2n+1}}{2n+1}$$

(c) Find, in simplest terms, the coefficient of x^5.

(d) Use your approximation up to and including the term in x^5 to find an approximate value for $arsinh\ 0.5$.

(e) Calculate the percentage error in using this approximation.

Ex 6D pg 133

Standard Integrals

Same as non-hyperbolic version?

✗ $\int \sinh x \ dx = \cosh x + C$

✓ $\int \cosh x \ dx = \sinh x + C$

Not in this chapter but worth briefly mentioning. (covers the next four integrals)

✓ $\int \operatorname{sech}^2 x \ dx = \tanh x + C$

Not in formula booklet. (covers the next three integrals)

✓ $\int \operatorname{cosech}^2 x \ dx = -\coth x + C$

✗ $\int \operatorname{sech} x \tanh x \ dx = -\operatorname{sech} x + C$

✓ $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$

Was covered in Chapter 3. (covers the next two integrals)

$$\int \frac{1}{\sqrt{1-x^2}} \ dx = \arcsin x + C, \qquad |x| < 1$$

$$\int \frac{1}{1+x^2} \ dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \ dx = \operatorname{arcsin}h\, x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} \ dx = \operatorname{arccosh} x + C, \qquad x > 1$$

Standard Integral Examples

$$\int \cosh(4x-1)\, dx =$$

$$\int sinh\left(\frac{2}{3}x\right)\ dx =$$

$$\int \frac{3}{\sqrt{1+x^2}} dx =$$

$$\int \frac{4}{\sqrt{x^2-1}} dx =$$

$$\int sinh\,(3x)\ dx =$$

$$\int \frac{10}{\sqrt{x^2-1}} dx =$$

$$\int \frac{2}{\sqrt{1+x^2}} dx =$$

Not Quite so Standard Examples

1. $\int \frac{2+5x}{\sqrt{x^2+1}}\ dx$

2. $\int \cosh^5 2x \sinh 2x\ \ dx$

3. $\int \tanh x\ \ dx$

Using Identities

1. $\int \cosh^2 3x \ dx$

2. $\int \sinh^3 x \ dx$

What now??

Sometimes there are techniques which work on non-hyperbolic trig functions but doesn't work on hyperbolic ones. Just first replace any hyperbolic functions with their definition.

Examples

1. Find $\int e^{2x} \sinh x \ dx$

2. Find $\int \operatorname{sech} x \ dx$

Dealing with $1/\sqrt{a^2+x^2}$, $1/\sqrt{x^2-a^2}$, ...

We can use substitution to deal with this style of integration.

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \sinh^2 u = \cosh^2 u$$

- Consider $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$. What substitution might we use?

- Consider $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$. What substitution might we use?

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx =$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx =$$

Example

1. Show that $\int \frac{1}{\sqrt{x^2-a^2}} dx = arcosh\left(\frac{x}{a}\right) + c$

2. Show that $\int_5^8 \frac{1}{\sqrt{x^2-16}} dx = \ln\left(\frac{2+\sqrt{3}}{2}\right)$

3. Show that $\int \sqrt{1+x^2}\, dx = \frac{1}{2} arsinh\, x + \frac{1}{2} x\sqrt{1+x^2} + C.$

Test Your Understanding

[June 2013 Q2]

(*a*) Find

$$\int \frac{1}{\sqrt{(4x^2+9)}}\,dx$$

(2)

(*b*) Use your answer to part (*a*) to find the exact value of

$$\int_{-3}^{3} \frac{1}{\sqrt{(4x^2+9)}}\,dx$$

giving your answer in the form $k\ln(a+b\sqrt{5})$, where a and b are integers and k is a constant.

(3)

2)Using a hyperbolic substitution, evaluate $\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\,dx$

Integrating by Completing the Square

By completing the square, we can then use one of the standard results.

Examples

1. Determine $\int \frac{1}{x^2-8x+8} dx$

2. Determine $\int \frac{1}{\sqrt{12x+2x^2}} dx$

Test Your Understanding

[June 2014(R) Q2]

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c$$

(*a*) Find the values of the constants *a*, *b* and *c*. **(3)**

Hence, or otherwise, find

(*b*) $\displaystyle\int \frac{1}{9x^2 + 6x + 5}\,\mathrm{d}x$ **(2)**

(*c*) $\displaystyle\int \frac{1}{\sqrt{9x^2 + 6x + 5}}\,\mathrm{d}x$ **(2)**

Ex 6E and Mixed Ex 6