CP2, Chapter 3

Methods in Calculus

Course Structure

1. Improper Integrals
2. Mean Value of a Function
3. Differentiating and Integrating Inverse Trig Functions
4. Integrating using Partial Fractions

5 **Further calculus**	5.1	**Derive formulae for and calculate volumes of revolution.**	**Both** $\pi \int y^2 \, dx$ **and** $\pi \int x^2 \, dy$ **are required.** Students should be able to find a volume of revolution given either Cartesian equations or parametric equations.
	5.2	Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.	For example, $\int_0^\infty e^{-x} \, dx$, $\int_0^2 \frac{1}{\sqrt{x}} \, dx$
	5.3	Understand and evaluate the mean value of a function.	Students should be familiar with the mean value of a function f(x) as, $\frac{1}{b-a}\int_a^b f(x) \, dx$
	5.4	Integrate using partial fractions.	Extend to quadratic factors $ax^2 + c$ in the denominator

5 **Further calculus** *continued*	5.5	Differentiate inverse trigonometric functions.	For example, students should be able to differentiate expressions such as, $\arcsin x + x\sqrt{(1-x^2)}$ and $\frac{1}{2}\arctan x^2$
	5.6	Integrate functions of the form $(a^2-x^2)^{-\frac{1}{2}}$ and $(a^2-x^2)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.	

<u>Improper Integrals</u>

STARTER 1: Determine $\int_{-1}^{1} \frac{1}{x^2}\, dx$. Is there an issue?

STARTER 2: Determine $\int_{-1}^{1} \frac{1}{x^2}\, dx$. Is there an issue?

STARTER 3: Determine $\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx$. Is there an issue?

If a function $f(x)$ exists and is continuous for all values of x in the interval $[a, b]$ then the definite integral $\int_a^b f(x)\, dx$ represents the area enclosed by the curve $y = f(x)$, the x axis and the lines $x = a$ and $x = b$.

Here, we consider integrals where one or both of the limits are infinite, or where the function is not defined at some point within in the given interval. These are called improper integrals. In these cases, it is still possible for the function to enclose a finite area.

The integral $\int_a^b f(x)\,dx$ is improper if either:

- One or both of the limits is infinite
- $f(x)$ is undefined at $x = a,\ x = b$ are another point in the interval $[a, b]$.

If an improper integral exists it is said to be **convergent.** If it does not exist it is said to be **divergent.**

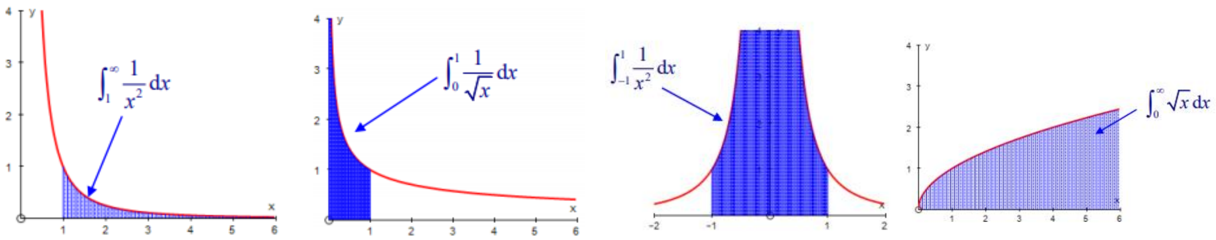
For example:

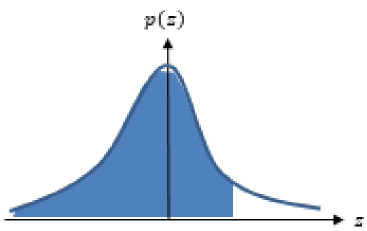
$\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ is an improper integral since one of its limits is infinity;

$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is an improper integral since it is undefined at $x = 0$.

$\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x$ is an improper integral since it is undefined at $x = 0$;

$\int_0^{\infty} \sqrt{x}\,\mathrm{d}x$ is an improper integral since one of its limits is infinity;





We've seen use of improper integrals with the **normal distribution** (Stats Year 2), which has the probability function:

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}}$$

The z table determines the probability up to a particular value of z. So $\phi(z) = P(Z \le z) = \int_{-\infty}^{z} p(x)\,dx$. As the area under the whole graph is 1, the improper integral $\int_{-\infty}^{\infty} p(x)\,dx = 1$

We can't use ∞ in calculations directly. We can make use of the lim function we saw in differentiation by first principles.

To find $\int_a^{\infty} f(x)\,dx$, determine $\lim_{t\to\infty} \int_a^t f(x)\,dx$

Examples

1. Evaluate $\int_1^{\infty} \frac{1}{x^2}\,dx$ or show that it is not convergent.

2. Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$ or show that it is not convergent.

Undefined Values of $f(x)$

We need to **avoid values** with the range $[a, b]$ **for which the expression is not defined**. But just as we avoided ∞ by considering the limit as $t \to \infty$, we can similarly find what the area converges to as x tends towards the undefined value.

Examples

1. Evaluate $\int_0^1 \frac{1}{x^2}\, dx$ or show that it is not convergent.

2. Evaluate $\int_0^2 \frac{x}{\sqrt{4-x^2}}\, dx$ or show that it is not convergent.

Further Examples

Find, if possible, the values of

(i) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(ii) $\int_{-1}^1 \frac{1}{x^2}\,dx$

Integrating between $-\infty$ and ∞

If both limits in the integral are infinite, then you need to split the integral into the sum of 2 improper integrals.

If both these integrals converge, then the original integral converges, but if either diverges, then the original integral is also divergent.

Example

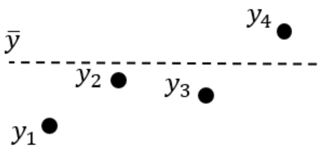
(a) Find $\int xe^{-x^2} dx$

(b) Hence show that $\int_{-\infty}^{\infty} xe^{-x^2}\, dx$ converges and find its value.

Ex 3a, pages 56-58

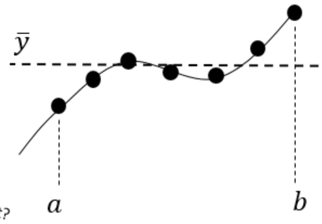
The Mean Value of a Function

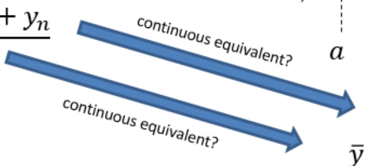
How would we find the mean of a set of values y values $y_1, y_2, \ldots, y_n$?



$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

So the question then is, can we extend this to the continuous world, with a function $y = f(x)$, between $x = a$ and $x = b$?





$$\bar{y} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Integration can be thought of as the continuous version of summation of the y values.

The width of the interval, $b - a$, could (sort of) be thought of as the number of points in the interval on an infinitesimally small scale.

✎ The **mean value** of the function $y = f(x)$ over the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_b^a f(x)dx$$

We write it as $\bar{y}$ or $\bar{f}$ or y_m.

Textbook Example

1. Find the mean value of $f(x) = \frac{4}{\sqrt{2+3x}}$ over the interval $[2,6]$.

2. $f(x) = \frac{4}{1+e^x}$

(a) Show that the mean value of $f(x)$ over the interval $[\ln 2, \ln 6]$ is $\frac{4\ln\frac{9}{7}}{\ln 3}$

(b) Use your answer to part a to find the mean value over the interval $[\ln 2, \ln 6]$ of $f(x) + 4$.

Use geometric considerations to write down the mean value of $-f(x)$ over the interval $[\ln 2, \ln 6]$

Differentiating inverse trigonometric functions

Examples

1. Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Remember that we're trying to turn $\cos y$ into an expression in terms of x; we have to use $x = \sin y$ in some way. You then might think "Oh, I know an identity that relates $\cos y$ and $\sin y$!"

2. Given that $y = \arcsin x^2$ find $\frac{dy}{dx}$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$$

<u>Test Your Understanding</u>

1. Given that $y = \operatorname{arcsec} 2x$, show that $y = \frac{1}{x\sqrt{4x^2-1}}$

2. Given that $y = \arctan\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$

Ex 3C Pg 62

Integrating with inverse trigonometric functions

Use an appropriate substitution to show that $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

Think what value of x would make $1 + x^2$ nicely simplify.

Dealing with $1/(a^2 - x^2)$, $1/\sqrt{a^2 - x^2}$, ….

Use our previous results to fill in the table below:

$\int \frac{1}{\sqrt{1-x^2}}\,dx =$	$\lvert x\rvert < a$
$\int \frac{1}{1+x^2}\,dx =$	

Show that $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$ where a is a positive constant and $\lvert x\rvert < a$.

We can extend these results to give some standard results which are given in the formulae book:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C, \ \ |x| < a$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

1. Find $\int \frac{4}{5+x^2} dx$

2. Find $\int \frac{1}{25+9x^2} dx$

3. Find $\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}} dx$

4. Find $\int \frac{x+4}{\sqrt{1-4x^2}} dx$

Ex 3D Pg 67-68

Solving using partial fractions

We have already seen in Pure Year 2 how we can use partial fractions to integrate. We can use this to further expand our repertoire of integration techniques for expressions of the form $\frac{1}{a^2 \pm x^2}$ and $\frac{1}{\sqrt{a^2 \pm x^2}}$

Example

Prove that $\int \frac{1}{a^2 - x^2}\ dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$

When you write as partial fractions, ensure you have the **most general possible non-top heavy fraction**, i.e. the 'order' (i.e. maximum power) of the numerator is **one less** than the denominator.

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Example

Show that $\int \frac{1+x}{x^3+9x}\,dx = A\ln\left(\frac{x^2}{x^2+9}\right) + B\arctan\left(\frac{x}{3}\right) + c$, where A and B are constants to be found.

If the fraction is top-heavy, you'll have a quotient. As per Pure Year 2, if the order of numerator and denominator is the same, you'll need an extra constant term. If the power is 1 greater in the numerator, you'll need a quotient of $Ax + B$, and so on.

$$\frac{4x^2 + x}{x^2 + x} = \frac{4x^2 + x}{x(x+1)} = A + \frac{B}{x} + \frac{C}{x+1}$$

Test Your Understanding

(a) Express $\frac{x^4+x}{x^4+5x^2+6}$ as partial fractions.

(b) Hence find $\int \frac{x^4+x}{x^4+5x^2+6}\ dx.$

Ex 3D Pg 67-68