# CP1 Chapter 8

# **Proof by Induction**

# **Chapter Overview**

- 1. Summation Proofs
- 2. Divisibility Proofs
- 3. Matrix proofs

Торіс	What students need to learn:						
	Conte	nt	Guidance				
1 Proof	1.1	Construct proofs using mathematical induction. Contexts include sums of series, divisibility and powers of matrices.	To include induction proofs for  (i) summation of series  e.g. show $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ or  show $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ (ii) divisibility e.g. show $3^{2n} + 11$ is divisible by 4  (iii) matrix products e.g. show $ \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix}$				

### What is poof by induction?

We can often use **proof by induction** whenever we want to show some property holds for all integers (usually positive) up to infinity.

#### Example

Show that 
$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$
 for all  $n \in \mathbb{N}$ .

We can show this is true for certain examples. However, no number of examples are sufficient to show the statement is true for **all** integers n.

The key part of induction is that we need to show that "if the statement is true for a particular value, then it is true for the next value".

#### Basic Example:

Prove by induction that if the first term of a sequence is  $a_1=3$  and  $a_{n+1}=a_n+2$ , then every term in the sequence is odd.

#### **Proof by induction:**

Step 1: **Basis:** Prove the general statement is true for n = 1.

Step 2: **Assumption:** Assume the general statement is true for n = k.

Step 3: **Inductive:** Show that the general statement is then true for n = k + 1.

Step 4: **Conclusion:** The general statement is then true for all positive integers n.

### Type 1: Summation Proofs

Example 1: Show that  $\sum_{r=1}^{n} (2r-1) = n^2$  for all  $n \in \mathbb{N}$ .

Basis step:

Assumption:

Inductive:

Conclusion:

## Conclusion step – key points

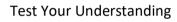
(For method mark)

Any 3 of these seen anywhere in the proof:

- "true for n=1"
- "assume true for n = k"
- "true for n = k + 1"
- "true for all n/positive integers"

Example 2. Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r 2^{r} = 2(1 + (n-1)2^{n})$$



8. (a) Prove by induction that, for any positive integer n,

$$\sum_{r=1}^{n} r^{3} = \frac{1}{4} n^{2} (n+1)^{2}.$$

**(5)** 

# Type 2: Divisibility Proofs

There are different techniques for the induct consider 2 different methods.	ive step. For the following example we will
Example	
Prove by induction that $3^{2n} + 11$ is divisible	by 4 for all positive integers $n$ .
Basis step:	
Assumption:	
Inductive step: method 1	Inductive step: method 2

Conclusion:

## Example:

Prove by induction that  $8^n - 3^n$  is divisible by 5.

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Prove by induction that  $n^3 - 7n + 9$  is divisible by 3 for all positive integers n.

## Type 3: Matrix Proofs

Example

Prove by induction that 
$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$$
 for all  $n \in \mathbb{Z}^+$ .

## Example

Prove by induction that  $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$  for all  $n \in \mathbb{Z}^+$ .

#### Be the examiner!

How many marks does this deserve??

**6.** Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

Let 
$$f(x) = 3^{2(1)k} 4 - 2^{4(1)} = 3^{6} - 2^{2} = 729 - 4 = 725 = 145 \times 5$$

if  $f(n)$  divisible by 5 when  $n = 1$ 

Assume trace for  $n = k$ :

$$5 | f(k) = 3 | f(k) = 3^{2k+4} = 2^{2k} = 5a \quad \text{where} \quad a \in \mathbb{N}$$

Consider  $f(k+1) : m$ 

$$f(k+1) = 3^{2(k+1)+4} - 2^{2(k+1)} = 3^{2k+6} - 2^{2k+2} = 9(3^{2k+4}) - 4(2^{2k})$$

$$= f(k+1) = 9(3^{2k+4} - 2^{2k}) + 5(2^{2k}) = 9f(k) + 5(2^{2k}) = 5(9a + 2^{2k})$$

within is divisible by 5 or  $f(k+1)$  obvisible by 5

Since true for n=1 => true for Vn E It by mathematical induction.