CP1 Chapter 8

Proof by Induction

Chapter Overview

1. Summation Proofs
2. Divisibility Proofs
3. Matrix proofs

Topic	What students need to learn:		
	Content		Guidance
1 Proof	1.1	Construct proofs using mathematical induction. Contexts include sums of series, divisibility and powers of matrices.	To include induction proofs for (i) summation of series e.g. show $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ or show $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ (ii) divisibility e.g. show $3^{2n}+11$ is divisible by 4 (iii) matrix products e.g. show $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix}$

What is poof by induction?

We can often use **proof by induction** whenever we want to show some property holds for all integers (usually positive) up to infinity.

Example

Show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

We can show this is true for certain examples. However, no number of examples are sufficient to show the statement is true for **all** integers n.

The key part of induction is that **we need to show that "if the statement is true for a particular value, then it is true for the next value**".

Basic Example:

Prove by induction that if the first term of a sequence is $a_1 = 3$ and $a_{n+1} = a_n + 2$, then **every term in the sequence is odd**.

Proof by induction:

Step 1: **Basis:** Prove the general statement is true for $n = 1$.

Step 2: **Assumption:** Assume the general statement is true for $n = k$.

Step 3: **Inductive:** Show that the general statement is then true for $n = k + 1$.

Step 4: **Conclusion:** The general statement is then true for all positive integers n.

Type 1: Summation Proofs

Example 1: Show that $\sum_{r=1}^{n}(2r - 1) = n^2$ for all $n \in \mathbb{N}$.

Basis step:

Assumption:

Inductive:

Conclusion:

Conclusion step – key points

(For method mark)

Any 3 of these seen anywhere in the proof:

- **"true for $n = 1$"**
- **"assume true for $n = k$"**
- **"true for $n = k + 1$"**
- **"true for all n/positive integers"**

Example 2. Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r2^r = 2(1 + (n-1)2^n)$$

Test Your Understanding

8. (*a*) Prove **by induction** that, for any positive integer n,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

(5)

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Type 2: Divisibility Proofs

There are different techniques for the inductive step. For the following example we will consider 2 different methods.

Example

Prove by induction that $3^{2n} + 11$ is divisible by 4 for all positive integers n.

Basis step:

Assumption:

Inductive step: method 1

Inductive step: method 2

Conclusion:

Example:

Prove by induction that $8^n - 3^n$ is divisible by 5.

Question:

Prove by induction that $n^3 - 7n + 9$ is divisible by 3 for all positive integers n.

Type 3: Matrix Proofs

Example

Prove by induction that $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 2^n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.

Example

Prove by induction that $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.

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Be the examiner!

How many marks does this deserve??

6. Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

Let ~~f~~ $n = 1$:

$f(1) = 3^{2(1)+4} - 2^{2(1)} = 3^6 - 2^2 = 729 - 4 = 725 = 145 \times 5$

$\therefore$ ~~f~~ $f(n)$ divisible by 5 when $n = 1$

Assume true for $n = k$:

$5 \mid f(k) \Rightarrow f(k) = 3^{2k+4} - 2^{2k} = 5a$ where $a \in \mathbb{N}$

Consider $f(k+1)$:

$f(k+1) = 3^{2(k+1)+4} - 2^{2(k+1)} = 3^{2k+6} - 2^{2k+2} = 9(3^{2k+4}) - 4(2^{2k})$

$\Rightarrow f(k+1) = 9(3^{2k+4} - 2^{2k}) + 5(2^{2k}) = 9f(k) + 5(2^{2k}) = 5(9a + 2^{2k})$

~~$\Rightarrow f(k+1)$ is divisible by 5~~

$\therefore f(k)$ divisible by 5 $\Rightarrow f(k+1)$ divisible by 5

Since true for $n = 1 \Rightarrow$ true for $\forall n \in \mathbb{Z}^+$ by mathematical induction.