CP1 Chapter 6

Matrices

Chapter Overview

1. Understand matrices and perform basic operations (adding, scalar multiplication)

2. Multiply Matrices

3. Find the determinant or inverse of a matrix

4. Solve simultaneous equations using matrices

3 **Matrices**	3.1	**Add, subtract and multiply conformable matrices.** **Multiply a matrix by a scalar.**	
	3.2	**Understand and use zero and identity matrices.**	

3 **Matrices** *continued*	3.3	**Use matrices to represent linear transformations in 2-D.** **Successive transformations.** **Single transformations in 3-D.**	**For 2-D, identification and use of the matrix representation of single and combined transformations from: reflection in coordinate axes and lines $y = \pm x$, rotation through any angle about $(0, 0)$, stretches parallel to the x-axis and y-axis, and enlargement about centre $(0, 0)$, with scale factor k, $(k \neq 0)$, where $k \in \mathbb{R}$.** **Knowledge that the transformation represented by AB is the transformation represented by B followed by the transformation represented by A.** **3-D transformations confined to reflection in one of $x = 0, y = 0, z = 0$ or rotation about one of the coordinate axes.** **Knowledge of 3-D vectors is assumed.**
	3.4	**Find invariant points and lines for a linear transformation.**	**For a given transformation, students should be able to find the coordinates of invariant points and the equations of invariant lines.**
	3.5	**Calculate determinants of** **2 x 2 and 3 x 3 matrices and interpret as scale factors, including the effect on orientation.**	**Idea of the determinant as an area scale factor in transformations.**
	3.6	**Understand and use singular and non-singular matrices.** **Properties of inverse matrices.** **Calculate and use the inverse of non-singular 2 x 2 matrices and 3 x 3 matrices.**	**Understanding the process of finding the inverse of a matrix is required.** **Students should be able to use a calculator to calculate the inverse of a matrix.**

3 **Matrices** *continued*	3.7	**Solve three linear simultaneous equations in three variables by use of the inverse matrix.**	
	3.8	**Interpret geometrically the solution and failure of solution of three simultaneous linear equations.**	**Students should be aware of the different possible geometrical configurations of three planes, including cases where the planes,** **(i) meet in a point** **(ii) form a sheaf** **(iii) form a prism or are otherwise inconsistent**

Introduction

A matrix (plural: matrices) is **simply an 'array' of numbers**, e.g.

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 3 & 0 \end{pmatrix}$$

On a simple level, a matrix is just a way to organise values into rows and columns, and represent these multiple values as a single structure.

The dimension of a matrix is its **size**, in terms of its number of **rows** and **columns** (in that order).

Examples:

Matrix	Dimension
$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$	
$\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$	
$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$	

Matrix Fundamentals

Operations with Matrices

1. Addition and subtraction

2. Scalar Multiplication

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3. Matrix Multiplication

Examples

1. $\begin{pmatrix} 2 & -4 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} =$

2.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 8 & 4 & 3 \\ 7 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 7 \\ 0 & 3 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

Matrix Multiplication Involving I:

Test Your Understanding

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2$

4. $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^k$

5. $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

6. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Matrices Grid Activity

When is Matrix Multiplication Valid?

Matrix multiplications are not always valid: the dimensions have to agree.

- For two matrices A and B, the matrix multiplication AB is valid provided A has the same number of columns as B has rows.
- If we multiply an n x m matrix by an m x k matrix we generate an n x k matrix.
- Note that only **square matrices** (i.e. same width as height) can be raised to a power.

Properties of Matrix Operations

Properties of Addition

The basic properties of addition for real numbers also hold true for matrices.

Let A, B and C be m x n matrices

A + B = B + A commutative

A + (B + C) = (A + B) + C associative

Properties of Multiplication

Let A, B and C be matrices of dimensions such that the following are defined. Then

A(BC) = (AB)C associative

A(B + C) = AB + AC distributive

(A + B)C = AC + BC distributive

But AB =/= BA non - commutative

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Matrices and their Inverses

Determinants

- The determinant of a matrix has many applications in matrices. Fundamentally, the determinant is required to find the inverse of a matrix.
- The determinant of a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det(\mathbf{A}) = |\mathbf{A}| = ad - bc$$

- If $det(\mathbf{A}) = 0$, then $\mathbf{A}$ is a **singular matrix** and it does not have an inverse.
- If $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is a **non-singular matrix** and it has an inverse.

Quickfire Questions:

$\mathbf{A}$	$\det(\mathbf{A})$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
$\begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix}$	
$\begin{pmatrix} 10 & -2 \\ 4 & -1 \end{pmatrix}$	

<u>Example</u>

The matrix $\mathbf{M} = \begin{pmatrix} 1-k & 2 \\ -1 & 4-k \end{pmatrix}$ is singular. Find the possible values of k.

Example

Given that $\mathbf{A}$ is singular, find the value of p.

$$A = \begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$$

Test Your Understanding

$$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}, \text{ where } a \text{ is real.}$$

(*a*) Find det $\mathbf{A}$ in terms of a.

(2)

(*b*) Show that the matrix $\mathbf{A}$ is non-singular for all values of a.

(3)

Determinants of 3x3 Matrices

To find the determinant and inverse of a 3 x 3 matrix, it is first necessary to define some new terms.

- **The minor of an element**

The minor of a particular element of a matrix is found by eliminating the row and column of that element and finding the determinant of the remaining matrix.
For a 3 x 3 matrix, the remaining matrix will be a 2 x 2 matrix.

- **The cofactors of an element**

The cofactor of an element is its minor multiplied by a multiple of (-1) in the following pattern (called the place signs)

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Example

Find the minors of the elements 0, -6 and 5 in the below matrix:

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$

Test Your Understanding

For the matrix $\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$

find the minors of each of the following elements:

(i) 2 (ii) 1 (iii) -2

Example

For the matrix

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$$

find the cofactors of each of the following elements:

(i) 4 (ii) 0 (iii) -4

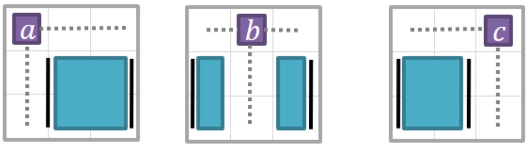
Finding the determinant

The determinant of a 3 x 3 matrix can be found from the cofactors of any row or column of the matrix. Each element in that row or column is multiplied by its cofactor, and the results are added together.

General Example:

(note the minus for the middle one)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



Example

$$\begin{vmatrix} 3 & 1 & 4 \\ 2 & 2 & 5 \\ -3 & 4 & 3 \end{vmatrix}$$

Test Your Understanding

1.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$ Determine $\det(\mathbf{A})$.

2.

$$\boldsymbol{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$$ where k is a constant.

Given that A is singular, find the possible values of k.

Alternative Method:

<u>Inverting a 2 x 2 Matrix</u>

Examples

1. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}^{-1}$

2. $\begin{pmatrix} 6 & 1 \\ 3 & 4 \end{pmatrix}^{-1}$

3. $\begin{pmatrix} 7 & 2 \\ 1 & -3 \end{pmatrix}^{-1}$

Test Your Understanding

1. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1}$

2. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$

3. For what value of p is $\begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$ singular? Given p is not this value, find the inverse.

Matrix Proofs

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Inverting a 3 x 3 Matrix

Example

Find the inverse of the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 1 & 4 & -2 \\ 2 & 1 & 1 \end{pmatrix}$

Using your Calculator

1. Mode → Matrix.
2. Select $MatA$. This allows you to input your matrix, which will be saved in a special variable '$MatA$'.
3. Select 3 rows/cols and input each number, pressing = after each.
4. Press AC to start a calculation.
5. You want to write $MatA^{-1}$. To get the $MatA$ in your expression: OPTN for the matrix menu, then select $MatA$ to insert it into your expression.
   Use the special x^{-1} key on your calculator, because the general power button will not work in matrix mode.
6. Press =, and look appropriately smug.

Further Example

$$\boldsymbol{A} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix},$$

and the matrix $\mathbf{B}$ is such that $(\mathbf{AB})^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}$.

(a) Show that $\mathbf{A}^{-1} = \mathbf{A}$.

(b) Find $\boldsymbol{B}^{-1}$.

Test Your Understanding

[June 2011 Q7] The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}, \quad k \neq 1.$$

(*a*) Show that det $\mathbf{M} = 2 - 2k$. **(2)**

(*b*) Find $\mathbf{M}^{-1}$, in terms of k. **(5)**

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Using Matrices for Simultaneous Equations

Examples

1. Use matrices to solve the set of linear equations

    1. $2x + 3y + z = 1$
    2. $\mathrm{x} + 2\mathrm{y} + \mathrm{z} = 2$
    3. $3\mathrm{x} + \mathrm{y} + \mathrm{z} = 0$

2. Use an inverse matrix to solve the simultaneous equations:

$$-x + 6y - 2z = 21$$
$$6x - 2y - z = -16$$
$$-2x + 3y + 5z = 24$$

Modelling Example

A colony of 1000 mole-rats is made up of adult males, adult females and youngsters. Originally there were 100 more adult females than adult males.

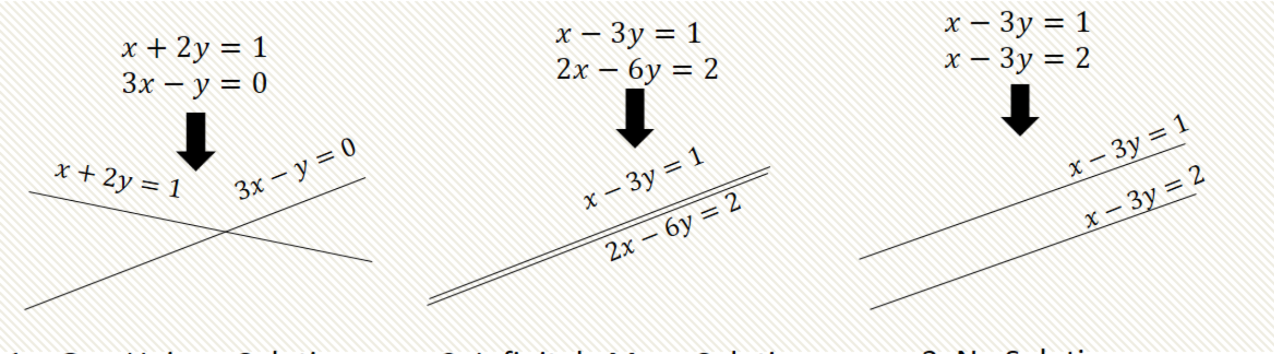
After one year:

- The number of adult males had increased by 2%
- The number of adult females had increased by 3%
- The number of youngsters had decreased by 4%
- The total number of mole-rats had decreased by 20

Form and solve a matrix equation to find out how many of each type of mole-rat were in the original colony.

Consistency of Linear Equations

As we know, the solution of a system of two equations (with two unknowns) can be visualised by finding the point of intersection of two lines.
A system of linear equations is known as consistent if there is at least one set of values that satisfies all the equations simultaneously (i.e. at least one point of intersection).



1. One Unique Solution

The system of equations is **consistent.** It has **one solution.**

The corresponding matrix $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ is **non-singular.**

$|A| \neq 0$

2. Infinitely Many Solutions

The system of equations is **consistent.** It has **infinitely many solutions.**

Matrix $\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}$ is **singular.**

$|A| = 0$

3. No Solution

The system of equations is **inconsistent.** It has **no solutions.**

Matrix $\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}$ is **singular.**

$|A| = 0$

Extending to 3 Variables

Again, we get solutions to the system of linear equations when all of the planes intersect.

Consider the possible outcomes for a set of 3 planes:

Scenario 1

Scenario 2

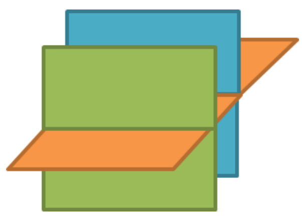
1.

2.

Scenario 3

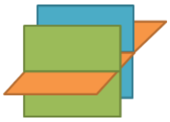
1.

2.



To classify solutions, we should:

1. First check for identical planes (equations which are equivalent) and therefore infinite solutions or parallel planes and therefore no solutions.



2. Next find the value of det A. If $|A| \neq 0$ the system of equations is consistent and there exists one unique solution.

3. If det A = 0 we have to check for parallel planes, either by sight (rows of matrix A are multiples) or by eliminating a variable and looking at the resulting linear equations…….

4. If the resulting 2d linear equations represent the same line then the original equations are consistent and therefore form a sheaf.

5. Otherwise, the planes form a prism and the system is inconsistent with no unique solution. (Parallel planes can be eliminated from the original equations)

Example

A system of equations is shown below:

$$3x - ky - 6z = k$$

$$kx + 3y + 3z = 2$$

$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 0$ (b) $k = 1$ (c) $k = -6$

Test Your Understanding

The system of equations is consistent and has a single solution. Determine the possible values of k.

$$2x + 3y - z = 13$$

$$3x - y + kz = 11$$

$$x + y + z = 7$$

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