Core Pure 1

Roots of Polynomials

Course Overview

1. Use relationships between coefficients and roots of a quadratic, cubic or quartic equation.

2. Find the value of expressions based on the roots of a polynomial.

3. Find the new polynomial when the roots undergo a linear transformation.

2.1	**Solve any quadratic equation with real coefficients.** **Solve cubic or quartic equations with real coefficients.**	**Given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics, for example:** **(i)** $f(z) = 2z^3 - 5z^2 + 7z + 10$ **Given that $2z - 3$ is a factor of $f(z)$, use algebra to solve $f(z) = 0$ completely.** **(ii)** $g(x) = x^4 - x^3 + 6x^2 + 14x - 20$ **Given $g(1) = 0$ and $g(-2) = 0$, use algebra to solve $g(x) = 0$ completely.**

Roots of Polynomials

The purpose of this chapter is to understand the underlying relationship between the **roots** of a polynomial, and the **coefficients** of each term.

Roots of Quadratics

If α and β are the roots of a quadratic $ax^2 + bx + c$ then

$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$

$=$

$=$

This pattern generalises to higher order polynomials which will be discussed further later on.

Quadratic Example

The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β. Without solving the equation, find the values of:

(a) $\alpha + \beta$

(b) $\alpha\beta$

(c) $\frac{1}{\alpha} + \frac{1}{\beta}$

(d) $\alpha^2 + \beta^2$

Question

The roots of the quadratic equation $3x^2 - 4x + 2 = 0$ are α and β. Without solving the equation, find the values of:

(a) $\alpha + \beta$

(b) $\alpha\beta$

(c) $\frac{1}{\alpha} + \frac{1}{\beta}$

(d) $\alpha^2 + \beta^2$

Example

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$. Find integer values for $a,\ b$ and c.

Test Your Understanding

1. **For the quadratic $x^2 + 2x + 3$, find:**

   (a) The sum of the roots.

   (b) The product of the roots.

2. **If the roots of a quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$ are $\boldsymbol{\alpha = \frac{2}{3}}$ and $\boldsymbol{\beta = \frac{1}{5}}$, determine integer values for $\boldsymbol{a, b, c}$.**

Ex 4a pg 56-57

Roots of Cubics

By the Fundamental Theorem of Algebra, a cubic equation $ax^3 + bx^2 + cx + d = 0$ always has 3 (potentially repeated) roots, α, β, γ. We saw in the previous chapters that these could be…

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Example

Find a cubic equation with roots 2, -1 and -3.

Example

1. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ **and** $\boldsymbol{\gamma}$ **are the roots of the cubic equation** $\mathbf{2x^3 + 3x^2 - 4x + 2 = 0}$**.**
Without solving the equation, find the values of:

(a) $\boldsymbol{\alpha + \beta + \gamma}$

(b) $\boldsymbol{\alpha\beta + \beta\gamma + \gamma\alpha}$

(c) $\boldsymbol{\alpha\beta\gamma}$

(d) $\boldsymbol{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}$

2. **The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i$, $\beta = 1 + 2i$ and $\gamma = 2$. Find integers values for a, b, c and d.**

Ex 4b pg 58-59

Roots of Quartics

Polynomial	Sum of roots	Sum of possible products of pairs of roots	Sum of products of triples	Sum of products of quadruples
Quadratic $ax^2 + bx + c$ (Roots: α, β)	$\alpha + \beta = -\frac{b}{a}$	$\alpha\beta = \frac{c}{a}$	N/A	N/A
Cubic $ax^3 + bx^2 + cx + d$ (Roots: α, β, γ)	$\alpha + \beta + \gamma = -\frac{b}{a}$	$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$	$\alpha\beta\gamma = -\frac{d}{a}$	N/A
Quartic $ax^4 + bx^3 + cx^2 + dx + e$ (Roots: $\alpha, \beta, \gamma, \delta$)	$\alpha + \cdots + \delta = -\frac{b}{a}$	$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$	$\alpha\beta\gamma + \alpha\beta\delta + \cdots = -\frac{d}{a}$	$\alpha\beta\gamma\delta = \frac{e}{a}$

Example

Find the quartic equation with roots 1, -2 and 3 (repeated).

Example

The equation $x^4 + 2x^3 + px^2 + qx - 60 = 0,\ x \in \mathbb{C},\ p, q \in \mathbb{R}$, has roots $\alpha,\ \beta, \gamma, \delta$. Given that $\gamma = -2 + 4i$ and $\delta = \gamma^*$.

(a) Show that $\alpha + \beta - 2 = 0$ and that $\alpha\beta + 3 = 0$

(b) Hence find all the roots of the quartic equation and find the values of p and q.

Ex 4C pg 60-61

Expressions Related to the Roots of Polynomials

Sums of squares:

- Quadratic: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- Cubic: $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
- Quartic: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\beta + \alpha\gamma + \beta\gamma + \cdots)$

Sums of cubes:

- Quadratic: $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- Cubic: $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$

Reciprocals:

- Quadratic: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$
- Cubic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
- Quartic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta}$

Products of Powers

- Quadratic: $\alpha^n + \beta^n = (\alpha\beta)^n$
- Cubic: $(\alpha + \beta + \gamma)^n = (\alpha\beta\gamma)^n$
- Quartic: $(\alpha + \beta + \gamma + \delta)^n = (\alpha\beta\gamma\delta)^n$

<u>Example</u>

The three roots of a cubic equation are $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. Given that $\boldsymbol{\alpha\beta\gamma = 4}$, $\boldsymbol{\alpha\beta + \beta\gamma + \gamma\alpha = -5}$ and $\boldsymbol{\alpha + \beta + \gamma = 3}$, find the value of

$\boldsymbol{(\alpha + 3)(\beta + 3)(\gamma + 3)}$

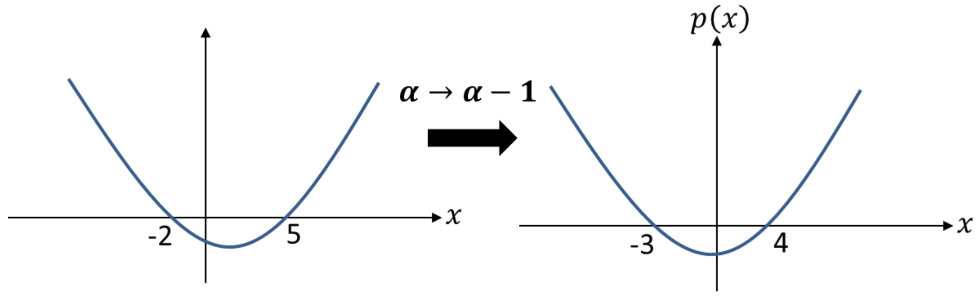
Ex 4D pg 63-64

Linear Transformations of Roots

We can use the general results regarding roots of polynomials to find an equation whose roots are related to the roots of the original equation i.e. the roots of the new equation are simply a linear transformation of the original roots.

Example

The polynomial $x^2 - 3x - 10 = 0$ has the roots α and β. Without finding the roots, determine the equation with roots $\alpha - 1$ and $\beta - 1$.



$p(x) = (x+2)(x-5)$
$= x^2 - 3x - 10$

$p(x) = (x+3)(x-4)$
$= x^2 - x - 12$

2 Methods:

Example

The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots $\alpha,\ \beta, \gamma, \delta$. Find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$.

Test Your Understanding

The cubic equation $x^3 - 2x^2 + 4 = 0$ has roots $\alpha,\ \beta, \gamma$. Find the equation with roots $(3\alpha - 1)$, $(3\beta - 1)$ and $(3\gamma - 1)$.

Ex 4E pg 66-67