Core Pure 1

Chapter 3: Series

Chapter Overview

1. Sum of 1's and Integers
2. Breaking down summations
3. Sum of Square and Cubes
4. Dealing with bounds

4.3	Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.	For example, students should be able to sum series such as $\sum_{r+1}^{n} r(r^2+2)$

Starter

Write out the following series:

1. $\sum_{r=1}^{n} r$

2. $\sum_{r=1}^{\infty} r$

3. $\sum_{r=1}^{n} (2r-1)$

4. $\sum_{r=0}^{n-1} (2r+1)$

5. $\sum_{r=1}^{n} (k^2+2)$

Calculate:

1. $\sum_{p=3}^{8} p^2$

2. $\sum_{r=0}^{5}(7r+1)^2$

The Sum of 1, n times

Example:

$$\sum_{r=1}^{n} 3$$

Test your Understanding:

1. $\sum_{r=1}^{6} 1$

2. $\sum_{r=1}^{7} 3$

3. $\sum_{r=4}^{n} 1$

<u>The sum of the first n natural numbers</u>

Examples:

Evaluate

$$1a. \sum_{r=1}^{100} r$$

$$1b. \sum_{r=50}^{100} r$$

Test Your Understanding:

Evaluate:

$$1a. \sum_{r=1}^{50} r$$

$$1b. \sum_{r=21}^{50} r$$

Further Examples:

1. $Evaluate \sum_{r=1}^{4}(2r-1)$

2. Show that

$$\sum_{r=5}^{2N-1} r = 2N^2 - N - 10$$

(for $N \geq 3$)

3. Show that $\sum_{r=k-1}^{2k} r = \frac{(k-2)(3k-1)}{2}$, $k \geq 1$

Test Your Understanding:

Show that $\sum_{r=n}^{3n} r = 2n(2n+1)$

Breaking up Summations

Prove that $\sum_{r=1}^{n} kr = k\sum_{r=1}^{n} r$, where k is a constant.

Examples:

1. $$\sum_{r=1}^{n} 3r$$

2. $$\sum_{r=1}^{n} 4$$

Prove that $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Combining the previous 2 statements leads to the following result:

$$\sum_{r=1}^{n}(ar + b) = a\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1$$

Examples

1. $\sum_{r=1}^{25}(3r + 1)$

2. Show that $\sum_{r=1}^{n}(3r+2) = \frac{n}{2}(3n+7)$

Hence evaluate $\sum_{r=20}^{50}(3r+2)$

Test Your Understanding

1. Show that $\sum_{r=1}^{n}(7r-4) = \left(\frac{n}{2}\right)(7n-1)$ and hence evaluate $\sum_{r=20}^{50}(7r-4)$

Exercise 3A Page 46-47

Sums of Squares and Cubes

Example

(a) Show that

$$\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$$

(b) Verify that the result is true for $n = 1$ and $n = 2$.

Example:

Find the sum of the following series

$$\sum_{r=1}^{n} r(r+3)(2r-1)$$

and hence evaluate

$$\sum_{r=11}^{40} r(r+3)(2r-1)$$

Test Your Understanding

5. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(r+2)(r+3) = \frac{1}{3}n(n^2+9n+26)$$

for all positive integers n.

(6)

(b) Hence show that

$$\sum_{r=n+1}^{3n}(r+2)(r+3) = \frac{2}{3}n(an^2+bn+c)$$

where a, b and c are integers to be found.

(4)

2.

(*a*) Use the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$, to prove that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+1)(n+2)(n+7)$$

for all positive integers n. **(5)**

(*b*) Hence, or otherwise, find the value of

$$\sum_{r=20}^{50} r(r+1)(r+5).$$

(2)

Extension: Given that n is even, determine $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \cdots - n^2$

Exercise 3B Page 46-47