Core Pure 1

Argand diagrams

Chapter Overview

1: Represent complex numbers on an Argand Diagram.

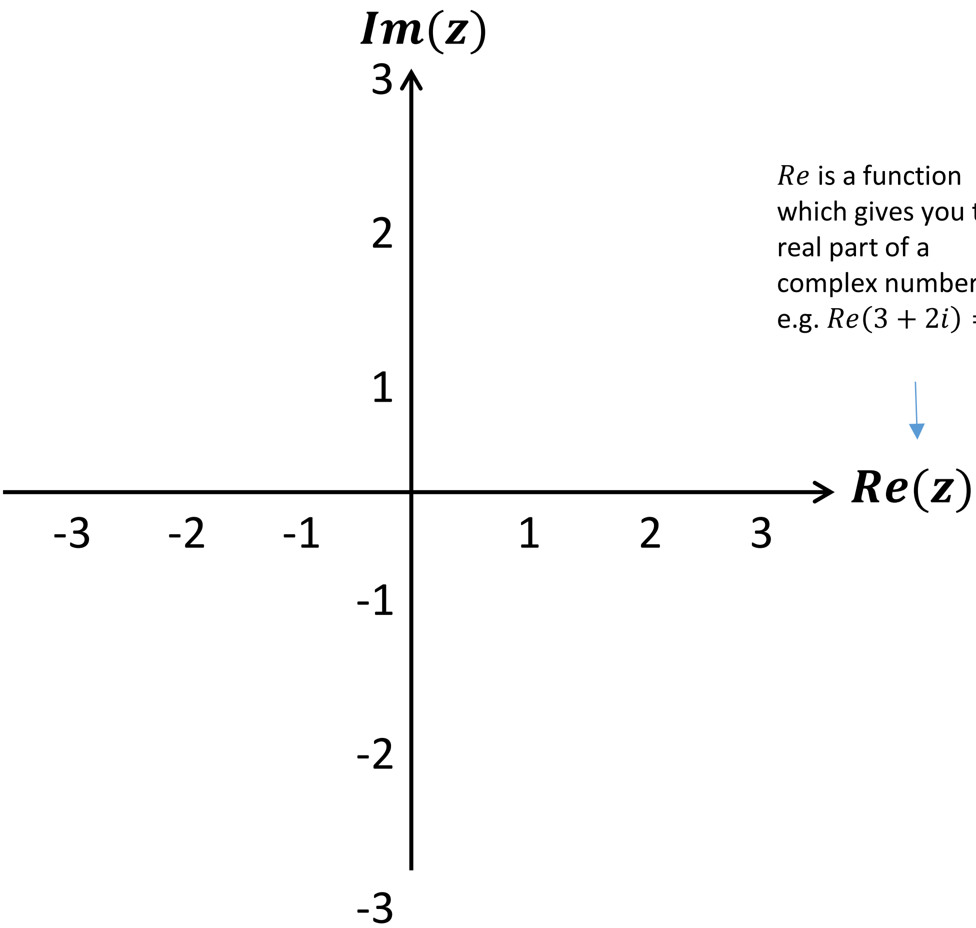
2: Put a complex number in modulus-argument form.

3: Identify loci and regions.

Topics	What students need to learn:		
	Content		Guidance
2 **Complex numbers** *continued*	2.4	Use and interpret Argand diagrams.	Students should be able to represent the sum or difference of two complex numbers on an Argand diagram.
	2.5	Convert between the Cartesian form and the modulus-argument form of a complex number. Knowledge of radians is assumed.	
	2.6	Multiply and divide complex numbers in modulus argument form. Knowledge of radians and compound angle formulae is assumed.	Knowledge of the results, $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \, \lvert z_2 \rvert$, $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
	2.7	Construct and interpret simple loci in the argand diagram such as $\lvert z - a \rvert > r$ and $\arg(z - a) = \theta$ Knowledge of radians is assumed.	To include loci such as $\lvert z - a \rvert = b$, $\lvert z - a \rvert = \lvert z - b \rvert$, $\arg(z - a) = \beta$, and regions such as $\lvert z - a \rvert \leqslant \lvert z - b \rvert$, $\lvert z - a \rvert \leqslant b$, $\alpha < \arg(z - a) < \beta$

Argand diagrams

Just as x-y axes were a useful way to visualise coordinates, an Argand diagram allows us to visualise complex numbers.



Re is a function which gives you the real part of a complex number. e.g. $Re(3+2i)=3$.

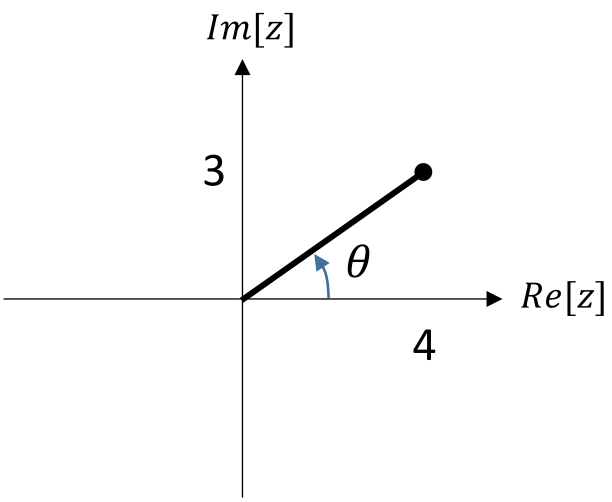
Exercise 2A Page 19

Modulus and argument

$4 + 3i$ is plotted on an Argand diagram.

a) What is its distance from the origin?

b) What is its anti-clockwise angle from the positive real axis? (in radians)



a)

b)

These are respectively known as the modulus $|z|$ and argument $\arg(z)$ of a complex number.

Examples

Determine the modulus and argument of:

(a) $5 + 12i$

(b) $-1 + i$

(c) $-2i$

(d) $-1 - 3i$

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$$z = 2 - 3i$$

(*a*) Show that $z^2 = -5 - 12i$. **(2)**

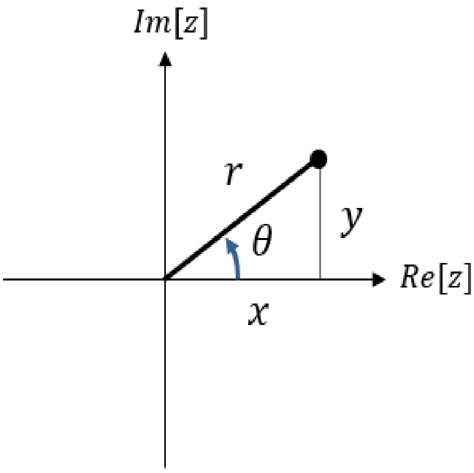
Find, showing your working,

(*b*) the value of $|z^2|$, **(2)**

(*c*) the value of arg (z^2), giving your answer in radians to 2 decimal places. **(2)**

(*d*) Show z and z^2 on a single Argand diagram. **(1)**

Modulus-Argument Form

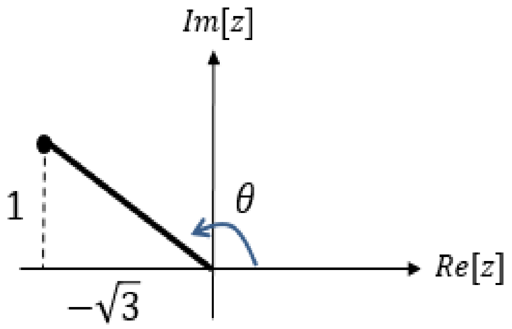


If we let $r = |z|$ and $\theta = \arg(z)$, can you think of a way of expressing z in terms of just r and θ?

Context: (r, θ) is known as a polar coordinate and you learn about these in Core Pure Year 2. Instead of coordinates being specified by their x and y position (known as a Cartesian coordinate), they are specified by their distance from the origin (the 'pole') and their rotation.

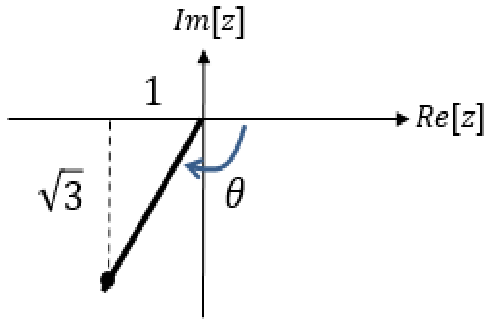
Example

Express $z = -\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$



Test Your Understanding

Express $z = -1 - \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$



Exercise 2C Page 24

Multiplying and Dividing Complex Numbers

Find the modulus and argument of $1 + i$ and $1 + \sqrt{3}$.
When you multiply these complex numbers together, do you notice anything about the modulus and argument of the result?

	$\mathbf{1+i}$	$\mathbf{1+i\sqrt{3}}$	$\mathbf{(1+i)(1+i\sqrt{3})}$ $=$
r			
θ			

Observation: The moduli have been multiplied, and the arguments have been added!

Proof (not assessed):

Recall from Pure Year 2 that:

$$sin(a+b) = sin\ a\ cos\ b + cos\ a\ sin\ bcos(a+b)$$
$$= cos\ a\ cos\ b - sin\ a\ sin\ b$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_1 = r_1(cos\ \theta_2 + i\ sin\ \theta_2)$.

Then $z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\ cos\ \theta_2 + sin\ \theta_1\cos\theta_2)]$

$$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

So $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

and $arg(z_1 z_2) = \theta_1 + \theta_2 = arg(z_1) + arg(z_2)$

Multiplying and Dividing Complex Numbers

Find the modulus and argument of $z_1 z_2$ where $z_1 = 2 + 3i$ and $z_2 = 1 + \sqrt{2}\, i$

Find the product of $z_1 = 5(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ and $z_2 = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

Express the following in the form x + iy

$z_1 = 4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$z_2 = 7(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3})$

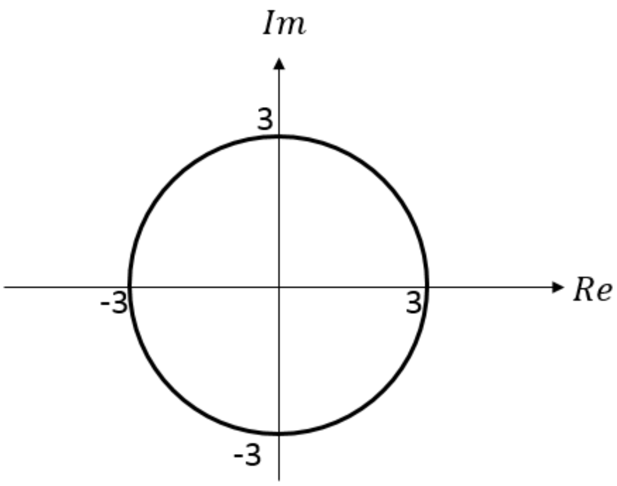
Remember: cos(-x) = cos(x) and sin(-x) = -sin(x)

[Textbook] Express $\frac{\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)}$ in the form $x+iy$

Loci

You have already encountered loci at GCSE as **a set of points (possibly forming a line or region) which satisfy some restriction**.

The definition of a circle for example is "a set of points equidistant from a fixed centre".



$|z| = 3$ means that the modulus of the complex number has to be 3. What points does this give us on the Argand diagram?

A quick reminder…

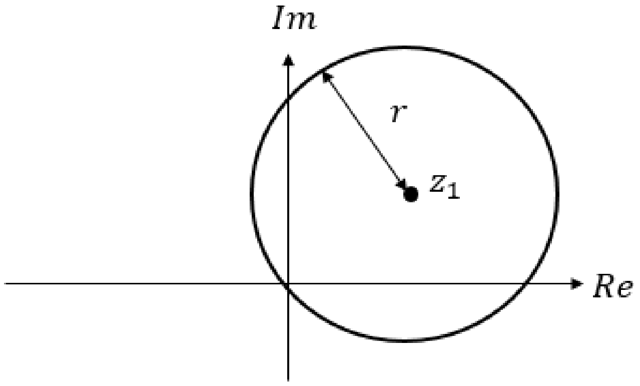
$$|z| = |\boldsymbol{x} + \boldsymbol{iy}|$$

$$=$$

$$|z - 3| = |\boldsymbol{x} + \boldsymbol{iy} - \boldsymbol{3}|$$

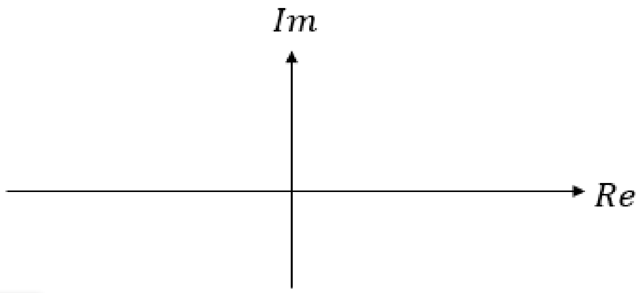
$$=$$

Loci of form $|z - z_1| = r$



What does $|z - z_1| = r$ mean?

Sketch the locus of points represented by $|z - 5 - 3i| = 3$



Find the Cartesian equation of this locus.

Questions:

Try Page 34 , Ex 2E Q1

Q2: Worked Example:

2 Given that z satisfies $|z - 5 - 4\mathrm{i}| = 8$,

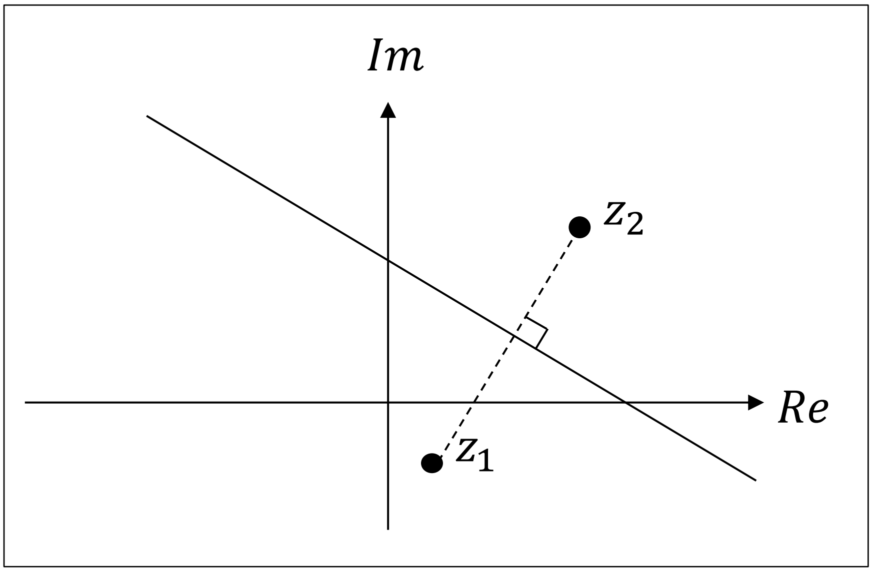
a sketch the locus of z on an Argand diagram

b find the exact values of z that satisfy:

i both $|z - 5 - 4\mathrm{i}| = 8$ and $\mathrm{Re}(z) = 0$ **ii** both $|z - 5 - 4\mathrm{i}| = 8$ and $\mathrm{Im}(z) = 0$

Loci of form $|z - z_1| = |z - z_2|$

What does $|z - z_1| = |z - z_2|$ mean?



Example,

Sketch the locus of points represented by $|z| = |z - 6i|$.
Write its equation.

Test Your Understanding So Far

Find the Cartesian equation of the locus of z if $|z-3| = |z+i|$, and sketch the locus of z on an Argand diagram.

What if we also required that $Re(z) = 0$?

Minimising/Maximising $\arg(z)$ and $|z|$

A complex number z is represented by the point P. Given that $|z - 5 - 3i| = 3$

(a) Sketch the locus of P

(b) Find the Cartesian equation of the locus.

(c) Find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$

(d) Find the minimum and maximum values of $|z|$

We did this earlier…

a)

b)

c)

d)

Quickfire Test Your Understanding

Given that the complex number z satisfies the equation $|z - 12 - 5i| = 3$, find the minimum value of $|z|$ and the maximum.

Minimising $|z|$ with perpendicular bisectors

(From earlier) Find the Cartesian equation of the locus of z if $|z - 3| = |z + i|$, and sketch the locus of z on an Argand diagram.

Hence, find the least possible value of $|z|$.

$\arg(z - z_1) = \theta$

$\arg(z) = \frac{\pi}{6}$?

$\arg(z + 3 + 2i) = \frac{3\pi}{4}$?

Regions

How would you describe each of the following in words? Therefore draw each of the regions on an Argand diagram.

$|z-4-2i| \leq 2$

$|z-4| < |z-6|$

$0 \leq \arg(z-2-2i) \leq \dfrac{\pi}{4}$

$|z-4-2i| \leq 2$, $|z-4| < |z-6|$ and $0 \leq \arg(z-2-2i) \leq \frac{\pi}{4}$

Test Your Understanding

P6 June 2003 Q4(i)(b)

Shade the region for which

$|z-1| \leq 1$ and $\frac{\pi}{12} \leq \arg(z+1) \leq \frac{\pi}{2}$