7) Methods in differential equations

7.1) First-order differential equations

Chapter CONTENTS

Worked example

Find general solutions to:

$$\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = -3$$

$$y = -3x + c$$

Worked example

Find general solutions to:

$$\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = 4x^3$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = 2x$$

$$y = x^2 + c$$

Worked example

Find general solutions to:

$$\frac{dy}{dx} = \frac{4y}{x}$$

$$\frac{dy}{dx} = \frac{3y}{x}$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$y = Ax^2$$

Worked example

Find general solutions to:

$$\frac{dy}{dx} = \sin x$$

$$\frac{dy}{dx} = \sec^2 x$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = \cos x$$

$$y = \sin x + c$$

Worked example

Find general solutions to:

$$\frac{dy}{dx} = y \tan x$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = y \cot x \, , 0 < x < \pi$$

$$y = A \sin x$$

Worked example

Find general solutions to:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$y = \pm\frac{A}{x}$, where $A = e^c$

Worked example

Find general solutions to:

$$\frac{dy}{dx} = xy + y$$

$$\frac{dy}{dx} = xy - x$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} = xy + x$$

$$y = Ae^{\frac{1}{2}x^2} - 1$$

Worked example	Your turn
Express as the derivative of one product:	Express as the derivative of one product:
$x^2 \frac{dy}{dx} + 2xy$	$x^3 \frac{dy}{dx} + 3x^2 y$
	$\frac{d}{dx}(x^3 y)$
$(\ln x) \frac{dy}{dx} + \frac{y}{x}$	$e^x \frac{dy}{dx} + e^x y$
	$\frac{d}{dx}(e^x y)$
$cos(x) \frac{dy}{dx} - y\, sin(x)$	$\sin(x) \frac{dy}{dx} + y \cos(x)$
	$\frac{d}{dx}(y \sin x)$

Worked example

Find general solutions to:

$$x^4 \frac{dy}{dx} + 4x^3 y = \cos x$$

Your turn

Find the general solution to:

$$x^3 \frac{dy}{dx} + 3x^2 y = \sin x$$

$$y = -\frac{1}{x^3}\cos x + \frac{c}{x^3}$$

Worked example

Find general solutions to:

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = e^x$$

Your turn

Find the general solution to:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$

$$y = x(e^x + c)$$

Worked example

Find general solutions to:

$$8x^3y\frac{dy}{dx} + 12x^2y^2 = x^4$$

Your turn

Find the general solution to:

$$4xy\frac{dy}{dx} + 2y^2 = x^2$$

$$y^2 = \frac{1}{6}x^2 + \frac{c}{2x}$$

Worked example

Find general solutions to:

$$\frac{dy}{dx} + 2y = e^x$$

$$\frac{dy}{dx} - y = e^{2x}$$

Your turn

Find the general solution to:

$$\frac{dy}{dx} - 4y = e^x$$

$$y = -\frac{1}{3}e^x + ce^{4x}$$

Worked example

Find general solutions to:

$$\cos x \frac{dy}{dx} + y \sin x = 1$$

Your turn

Find the general solution to:

$$\cos x \frac{dy}{dx} + 2y \sin x = \cos^4 x$$

$$y = \cos^2 x \, (\sin x + c)$$

Worked example

Find the particular solution such that $y = 3$ when $x = \pi$:

$$\cos x \frac{dy}{dx} + y \sin x = 1$$

Your turn

Find the particular solution such that $y = 2$ when $x = 0$:

$$\cos x \frac{dy}{dx} + 2y \sin x = \cos^4 x$$

$$y = \cos^2 x \, (\sin x + 2)$$

Worked example

Find the particular solution such that $y = 2$ when $x = 0$:

$$\cos x \frac{dy}{dx} - y = 1, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Your turn

Find the particular solution such that $y = 2$ when $x = 0$:

$$\cos x \frac{dy}{dx} + y = 1, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$y = 1 + \frac{\cos x}{1 + \sin x}$$

Worked example

Find general solutions to:

$$x\frac{dy}{dx} + 3y = \frac{\ln x}{x}, \qquad x > 0$$

Your turn

Find the general solution to:

$$x\frac{dy}{dx} + 5y = \frac{\ln x}{x}, \qquad x > 0$$

$$y = \frac{\ln x}{4x} - \frac{1}{16x} + \frac{c}{x^5}$$

Worked example

Solve the differential equation, giving y in terms of x, where

$$x^4 \frac{dy}{dx} - x^3 y = 1$$

and $y = 1$ at $x = 1$

Your turn

Solve the differential equation, giving y in terms of x, where

$$x^3 \frac{dy}{dx} - x^2 y = 1$$

and $y = 1$ at $x = 1$

$$y = -\frac{1}{3x^2} + \frac{4x}{3}$$

7.2) Second-order homogenous differential equations

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Worked example

Find general solutions to:

$$\frac{d^2y}{dx^2} = 6x$$

$$\frac{d^2y}{dx^2} = 24x^2$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} = 12x$$

$$y = 2x^3 + Ax + B$$

Worked example

Find general solutions to:

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$$

$$3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

Your turn

Find the general solution to:

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$$

$$y = Ae^{-\frac{3}{2}x} + Be^{-x}$$

Worked example

Find general solutions to:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

$$y = (A + Bx)e^{3x}$$

Worked example

Find general solutions to:

$$\frac{d^2y}{dx^2} + 9y = 0$$

$$\frac{d^2y}{dx^2} + 25y = 0$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} + 16y = 0$$

$$y = A\cos 4x + B\sin 4x$$

Worked example

Find general solutions to:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$$

$$\frac{d^2y}{dx^2} + 25y = 0$$

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 34y = 0$$

Your turn

Find general solutions to:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 8y = 0$$

$$y = Ae^{-4x} + Be^{-2x}$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$y = (A + Bx)e^{-3x}$$

$$\frac{d^2y}{dx^2} + 9y = 0$$

$$y = A\cos 3x + B\sin 3x$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$$

$$y = e^{-3x}(A\cos x + B\sin x)$$

7.3) Second-order non-homogenous differential equations

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Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 4$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3$$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 4x$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3x$$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}x + \frac{5}{12}$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 4x^2$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3x^2$$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{x}$$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^{x}$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 7\sin 4x$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 13\sin 3x$$

$$y = Ae^{3x} + Be^{2x} - \frac{1}{6}\sin 3x + \frac{5}{6}\cos 3x$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-3x}$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

$$y = Ae^{3x} + Be^{2x} - xe^{2x}$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 2$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$$

$$y = A + Be^{2x} - \frac{3}{2}x$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = x^2 - 2x + 3$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = x^2 - 3x + 2$$

$$y = Ae^{4x} + Be^{x} + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{7}{32}$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 24x^2$$

$$y = A + Be^{-4x} + 2x^3 - \frac{3}{2}x^2 + \frac{3}{4}x$$

Worked example

Find the general solution to:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 1 = e^x$$

Your turn

Find the general solution to:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 1 = e^x$$

$$y = \left(A + Bx + \frac{1}{2}x^2\right)e^x$$

Worked example

Find the general solution to:

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 2\sin t - \cos t$$

Your turn

Find the general solution to:

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

$$x = Ae^{-3t} + Be^{-3t} + \frac{3}{10}\cos t + \frac{1}{10}\sin t$$

7.4) Using boundary conditions

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Worked example

Find y in terms of x, given that $\frac{d^2y}{dx^2} - y = 2e^{-x}$, and that $\frac{dy}{dx} = 0$ and $y = 0$ at $x = 0$.

Your turn

Find y in terms of x, given that $\frac{d^2y}{dx^2} - y = 2e^{x}$, and that $\frac{dy}{dx} = 0$ and $y = 0$ at $x = 0$.

$$y = -\frac{1}{2}e^{x} + \frac{1}{2}e^{-x} + xe^{x}$$

Worked example

Find y in terms of x, given that $\frac{d^2y}{dx^2} + 25y = 3\cos 5x$, and that $\frac{dy}{dx} = 5$ and $y = 0$ at $x = 0$.

Your turn

Find x in terms of t, given that $\frac{d^2y}{dx^2} + x = 3\sin 2t$, and that $\frac{dx}{dt} = 1$ and $x = 0$ at $t = 0$.

$$x = 3\sin t - \sin 2t$$

Worked example

Solve the differential equation

$$\frac{d^2y}{dx^2} + 16y = \sin 4x$$

subject to boundary conditions $y = 0, \frac{dy}{dx} = 0$ when $x = 0$

Your turn

Solve the differential equation

$$\frac{d^2y}{dx^2} + 9y = \sin 3x$$

subject to boundary conditions $y = 0, \frac{dy}{dx} = 0$ when $x = 0$

$$y = \frac{1}{18}\sin 3x - \frac{1}{6}x\cos 3x$$