7) Linear transformations

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7.1) Linear transformations in two dimensions

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Worked example	Your turn
Find matrices to represent these linear transformations.	Find matrices to represent these linear transformations.
a) $T: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 3y - x \\ 2x \end{pmatrix}$	a) $T: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2y + x \\ 3x \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$
b) $V: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x + 3y \end{pmatrix}$	b) $V: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -2y \\ 3x + y \end{pmatrix}$ $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$

Worked example	Your turn
A rectangle R has vertices $(2,1),(4,1),(4,2)\ and\ (2,2)$ Find the vertices of the image of R under the transformation given by the matrix $\boldsymbol{M}=\begin{pmatrix}1 & 3\\ 3 & -1\end{pmatrix}$. Sketch R and its image, R' on a coordinate grid.	A square has vertices $(1,1),(3,1),(3,3)$ and $(1,3)$ Find the vertices of the image of S under the transformation given by the matrix $\boldsymbol{M}=\begin{pmatrix}-1 & 2\\ 2 & 1\end{pmatrix}$. Sketch S and the image of S on a coordinate grid. $(1,3),(-1,7),(3,9),(5,5)$

Worked example	Your turn
Determine if the point (2, 5) is invariant under the transformation given by the matrix: $$\begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$$ $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$	Determine if the point (4, 6) is invariant under the transformation given by the matrix: $$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$ No

Worked example	Your turn
Determine whether $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has any lines of invariant points	Determine whether $\begin{pmatrix} 5 & -2 \\ 3 & -0.5 \end{pmatrix}$ has any lines of invariant points $y = 2x$

Worked example	Your turn
Show that the matrix $\begin{pmatrix} 2 & -5 \\ 4 & -3 \end{pmatrix}$ has no invariant points other than the origin	Show that the matrix $\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}$ has no invariant points other than the origin $$\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$ $4x - 3y = x$ -> $y = x$ $2x - 5y = y$ -> $y = \frac{1}{3}x$ $x = \frac{1}{3}x$ -> $x = 0, y = 0$ $\therefore$ $(0, 0)$ is the only invariant point

Worked example	Your turn
Find the invariant lines of the transformation given by $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$	Find the invariant lines of the transformation given by $\begin{pmatrix} 3 & 2 \\ 5 & 6 \end{pmatrix}$ $y = \frac{5}{2}x + c$ $y = -x$

7.2) Reflections and rotations

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Worked example	Your turn
Find a 2×2 matrix that represents: • A reflection in the y-axis.	Find a 2×2 matrix that represents: • A reflection in the x-axis. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
• A reflection in the line $y = x$	• A reflection in the line $y = -x$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Worked example	Your turn
Find a 2×2 matrix that represents: • Rotation $90°$ anticlockwise about the origin • Rotation $180°$ about the origin	Find a 2×2 matrix that represents: • Rotation $270°$ anticlockwise about the origin $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Worked example	Your turn
Find a 2×2 matrix that represents: • Rotation $90°$ anticlockwise about the origin • Rotation $180°$ about the origin	Find a 2×2 matrix that represents: • Rotation $270°$ anticlockwise about the origin $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Worked example	Your turn
Describe fully the transformation described by the matrix $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$	Describe fully the transformation described by the matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ Rotation 45° anticlockwise about the origin

Worked example	Your turn
For these transformations, state any invariant lines/points: • reflection in the line $y = -x$ • Rotation $90°$ anticlockwise about the origin	For these transformations, state any invariant lines/points: • reflection in the line $y = x$ Invariant lines: $y = x$ Any straight line with gradient -1 $(y = -x + k)$ Invariant points: All points on those lines • Rotation $180°$ about the origin Invariant lines: Any straight line through origin $(y = mx)$ Invariant points: $(0, 0)$

Worked example	Your turn
$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ U is the single geometrical transformation represented by the matrix P. Given that U maps the point with coordinates (a, b) onto the point with coordinates $(2a - 3, 1 - b)$, find the values of a and b	$P = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ U is the single geometrical transformation represented by the matrix P. Given that U maps the point with coordinates (a, b) onto the point with coordinates $(3 + 2a, b + 1)$, find the values of a and b $a = -2, b = 1$

7.3) Enlargements and stretches

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Worked example	Your turn
Describe the effect of the following matrices:	Describe the effect of the following matrices:
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$
	Stretch parallel to x-axis, scale factor 2 and Stretch parallel to y-axis, scale factor 3 (not an enlargement)
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	
$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$	

Worked example

A triangle T has vertices $(1,1)$, $(1,2)$ and $(2,2)$.

a) Find the vertices of the image of T under the transformation given by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

b) Sketch T and its image, T' on a coordinate grid.

c) Describe the geometric transformation.

Your turn

A triangle T has vertices $(1,1)$, $(1,2)$ and $(2,2)$.

a) Find the vertices of the image of T under the transformation given by the matrix $\boldsymbol{M} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

b) Sketch T and its image, T' on a coordinate grid.

c) Describe the geometric transformation.

a) $(3,2)$, $(3,4)$ and $(6,4)$

b) Sketch

c) The triangle has been stretched by a scale factor of 3 parallel to the x-axis and by a scale factor of 2 parallel to the y-axis

Worked example	Your turn
Shape B is transformed to shape C by the matrix $$A = \begin{pmatrix} 2 & -3 \\ -4 & 9 \end{pmatrix}.$$ Given that the area of C is 72 square units, find the area of B	Shape R is transformed to shape S by the matrix $$A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}.$$ Given that the area of S is 72 square units, find the area of R 18

7.4) Successive transformations

Worked example

Represent as a single matrix the transformation representing a reflection in the line $y = x$ followed by a stretch parallel to the x-axis by a factor of 4.

Represent as a single matrix the transformation representing a rotation $90°$ anticlockwise about the point $(0,0)$ followed by a reflection in the line x-axis.
What single transformation is this?

Your turn

Represent as a single matrix the transformation representing a reflection in the line $y = -x$ followed by a stretch parallel to the y-axis by a factor of 3.

$$\begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}$$

Represent as a single matrix the transformation representing a rotation $270°$ anticlockwise about the point $(0,0)$ followed by a reflection in the line y-axis.
What single transformation is this?

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Reflection in the line $y = -x$

Worked example	Your turn
The matrix R is given by $R = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ a) Find R^3 b) Describe the geometric transformation represented by R^3 c) Hence describe the geometric transformation represented by R d) Write down R^{900}	The matrix R is given by $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ a) Find R^2 b) Describe the geometric transformation represented by R^2 c) Hence describe the geometric transformation represented by R d) Write down R^8 a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ b) Rotation $90°$ anticlockwise about $(0, 0)$ c) Rotation $45°$ anticlockwise about $(0, 0)$ d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

7.5) Linear transformations in three dimensions

Worked example	Your turn
Find the matrix representing: • reflection in the plane $x = 0$ • reflection in the plane $y = 0$	Find the matrix representing: • reflection in the plane $z = 0$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Worked example

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Describe the transformation represented by $\mathbf{M}$.
(b) Find the image of the point with coordinates $(-1, 2, 3)$ under the transformation represented by $\mathbf{M}$.

Your turn

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) Describe the transformation represented by $\mathbf{M}$.
(b) Find the image of the point with coordinates $(-1, 2, 3)$ under the transformation represented by $\mathbf{M}$.

(a) Reflection in the plane $z = 0$
(b) $(-1, 2, -3)$

Worked example	Your turn
Find the matrix representing: • Rotation, angle θ, anticlockwise about the x-axis • Rotation, angle θ, anticlockwise about the y-axis	Find the matrix representing: • Rotation, angle θ, anticlockwise about the z-axis $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Worked example	Your turn
Find the matrix representing: • Rotation, angle $90°$, anticlockwise about the x-axis • Rotation, angle $180°$, anticlockwise about the z-axis	Find the matrix representing: • Rotation, angle $270°$, anticlockwise about the y-axis $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Worked example

$$\mathbf{M} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Describe the transformation represented by $\mathbf{M}$.
(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by $\mathbf{M}$.

Your turn

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe the transformation represented by $\mathbf{M}$.
(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by $\mathbf{M}$.

(a) Rotation 30° anticlockwise about the y-axis

(b) $\left(\frac{1-\sqrt{3}}{2}, -2, \frac{1+\sqrt{3}}{2}\right)$

7.6) The inverse of a linear transformation

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Worked example	Your turn
The triangle T has vertices at A, B and C. The matrix $M = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$ transforms T to the triangle T' with vertices at $A'(3,4)$, $B'(10,4)$ and $C'(-3,-4)$. Determine the coordinates of A, B and C.	The triangle T has vertices at A, B and C. The matrix $M = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$ transforms T to the triangle T' with vertices at $A'(4,3)$, $B'(4,10)$ and $C'(-4,-3)$. Determine the coordinates of A, B and C. $A\ (1,0)\quad B(2,4)\quad C(-1,0)$

Worked example

$$M = \begin{pmatrix} 5 & -2 \\ 4 & -3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

a) Find det M

b) Describe fully the single geometrical transformation represented by A

c) The transformation represented by A followed by the transformation represented by B is equivalent to the transformation represented by M. Find B

Your turn

$$M = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

a) Find det M

b) Describe fully the single geometrical transformation represented by A

c) The transformation represented by A followed by the transformation represented by B is equivalent to the transformation represented by M. Find B

a) -23

b) Rotation $90°$ anticlockwise about $(0, 0)$

c) $\begin{pmatrix} -4 & 3 \\ 5 & 2 \end{pmatrix}$

Worked example	Your turn
$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & -2 \end{pmatrix}$$ The point (a, b, c) is mapped onto $(-3, -2, 1)$ under M. Find the values of a, b and c	$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & 3 & -1 \end{pmatrix}$$ The point (a, b, c) is mapped onto $(3, 2, -1)$ under M. Find the values of a, b and c $a = 10, b = -6, c = -7$

Worked example	Your turn
$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
a) Find R^{-1}	a) Find R^{-1}
b) Explain this geometrically	b) Explain this geometrically
c) Find R^{7999}	c) Find R^{8001}
d) Find R^{8000}	d) Find R^{8002}

a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b) Reflection is self-inverse. The inverse of reflection in the line $y = -x$ is reflection in the line $y = -x$ again

c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$