6) Matrices

6.1) Introduction to matrices

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Worked example

Write down the size of the matrix:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Your turn

Write down the size of the matrix:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

2×3

$$\begin{pmatrix} 1 & 2 \end{pmatrix}$$

1×2

Worked example	Your turn
Find (where possible):	Find (where possible):

Worked example

Find (where possible):

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}0 & -2\\-3 & -3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$$

$$\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}-3\\4\end{pmatrix}$$

Your turn

Find (where possible):

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix}+\begin{pmatrix}0 & -2 & -3\\-4 & -4 & -6\\-7 & -8 & -8\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\3 & 4\\5 & 6\end{pmatrix}-\begin{pmatrix}7\\8\\9\end{pmatrix}$$

Not additively conformable

Worked example

Find:

$$5\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

$$-7\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$$

Your turn

Find:

$$7\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}$$

$$\begin{pmatrix}7 & 14\\ 21 & 28\\ 25 & 42\end{pmatrix}$$

Worked example	Your turn
Find the value of k: $$\begin{pmatrix}-5\\3k\end{pmatrix}+k\begin{pmatrix}2k\\2k\end{pmatrix}=\begin{pmatrix}3k\\20\end{pmatrix}$$	Find the value of k: $$\begin{pmatrix}-3\\k\end{pmatrix}+k\begin{pmatrix}2k\\2k\end{pmatrix}=\begin{pmatrix}k\\6\end{pmatrix}$$ $$k=\frac{3}{2}$$

Worked example	Your turn
Write down the 2×2 identity matrix	Write down the 4×4 identity matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
Write down the 3×3 identity matrix	

6.2) Matrix multiplication

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Worked example

Determine the size of the matrix AB given the dimensions of A and B

Dimensions of A	Dimensions of B	Dimensions of AB (if valid)
3×2	2×1	
3×2	2×4	
3×2	4×2	
3×4	4×2	
2×4	4×2	
2×4	2×4	
2×2	2×4	
2×2	2×2	

Your turn

Determine the size of the matrix AB given the dimensions of A and B

Dimensions of A	Dimensions of B	Dimensions of AB (if valid)
2×3	3×1	2×1
1×4	1×4	Not valid
1×4	4×1	1×1
2×5	3×4	Not valid
3×3	3×3	3×3

Worked example	Your turn
Find the product of these matrices where possible:	Find the product of these matrices where possible:

Worked example

Find the product of these matrices where possible:

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}5 & 6\\7 & 8\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\3 & 4\\5 & 6\end{pmatrix}\begin{pmatrix}7 & 8\\9 & 10\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\begin{pmatrix}7 & 8\\9 & 10\end{pmatrix}$$

$$\begin{pmatrix}7 & 8\\9 & 10\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$$

Your turn

Find the product of these matrices where possible:

$$\begin{pmatrix}1 & 2\\3 & 4\\5 & 6\end{pmatrix}\begin{pmatrix}7 & 8 & 9\\10 & 11 & 12\end{pmatrix}$$

$$\begin{pmatrix}27 & 30 & 33\\61 & 68 & 75\\95 & 106 & 117\end{pmatrix}$$

Worked example	Your turn
Find:	Find:

Worked example

Find:

$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\begin{pmatrix} 7 & 8 & 9 \end{pmatrix}$$

Your turn

Find:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Worked example

Find:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^3$$

Your turn

Find:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^4$$

$$\begin{pmatrix} 199 & 290 \\ 435 & 634 \end{pmatrix}$$

Worked example	Your turn
Find: $$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^2$$ $$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}^3$$	Find: $$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}^k$$ $$\begin{pmatrix} 1 & 0 \\ ck & 1 \end{pmatrix}$$

Worked example	Your turn
$$A = \begin{pmatrix} 1 & a & 1 \\ -1 & 2 & 0 \\ b & 0 & 3 \end{pmatrix}$$ Given that $A^2 = \begin{pmatrix} 6 & -9 & 4 \\ -3 & 7 & -1 \\ 8 & -6 & 11 \end{pmatrix}$, find the values of a and b	$$A = \begin{pmatrix} 1 & -1 & b \\ a & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$ Given that $A^2 = \begin{pmatrix} -4 & -3 & -8 \\ 9 & 1 & -6 \\ 4 & -1 & 7 \end{pmatrix}$, find the values of a and b $a = 3, b = -2$

Worked example	Your turn
$A = \begin{pmatrix} -2 \\ a \end{pmatrix}$ and $B = \begin{pmatrix} 1 & b \end{pmatrix}$. Given that $BA = (0)$ find AB in terms of a	$A = \begin{pmatrix} -1 \\ a \end{pmatrix}$ and $B = \begin{pmatrix} b & 2 \end{pmatrix}$. Given that $BA = (0)$ find AB in terms of a $AB = \begin{pmatrix} -2a & -2 \\ 2a^2 & 2a \end{pmatrix}$

Worked example	Your turn
Find:	Find:

Worked example

Find:

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\3 & 4\\5 & 6\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$$

Your turn

Find:

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix}$$

6.3) Determinants

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Worked example	Your turn
Calculate the determinant then decide if the matrix has an inverse.	Calculate the determinant then decide if the matrix has an inverse.
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -3 \\ -1 & -4 \end{pmatrix}$ -7 Yes
$\begin{pmatrix} 1 & -2 \\ -3 & -4 \end{pmatrix}$	$\begin{pmatrix} 10 & -2 \\ 5 & -1 \end{pmatrix}$ 0 No
$\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$	

Worked example	Your turn
$A = \begin{pmatrix} 3 & p-1 \\ -2 & 4-p \end{pmatrix}$ Given that $\mathbf{A}$ is singular, find the value of p.	$A = \begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$ Given that $\mathbf{A}$ is singular, find the value of p. $p = \frac{14}{3}$

Worked example	Your turn
$\begin{vmatrix} 3 & 1 & 4 \\ 7 & 2 & 5 \\ -3 & 4 & 3 \end{vmatrix}$	$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$
Find the minor of:	Find the minor of:
a) 2	a) 5 $\quad \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$
b) -3	b) 0 $\quad \begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} = 37$
c) 7	c) -6 $\quad \begin{vmatrix} 1 & 2 \\ -1 & 8 \end{vmatrix} = 10$

Worked example

Calculate the determinant:

$$\begin{vmatrix} 2 & 1 & 0 \\ 5 & 4 & -6 \\ 8 & -1 & 2 \end{vmatrix}$$

Your turn

Calculate the determinant:

$$\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{vmatrix}$$

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Worked example	Your turn
$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{pmatrix}$ where k is a constant. Given that A is singular, find the possible values of k	$\boldsymbol{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$ where k is a constant. Given that A is singular, find the possible values of k $k = -\frac{1}{2}, -9$

6.4) Inverting a 2 x 2 matrix

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Worked example	Your turn
Find the inverse matrix for:	Find the inverse matrix for
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$
	$\frac{1}{2}\begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$ or $\begin{pmatrix} -2 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
$\begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$	

Worked example	Your turn
For what value of p is $\begin{pmatrix} 1 & 2-p \\ -4 & p+3 \end{pmatrix}$ singular?	For what value of p is $\begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$ singular? $p = \frac{14}{3}$
Given p is not this value, find the inverse.	Given p is not this value, find the inverse. $\frac{1}{14-3p}\begin{pmatrix} 3-p & -(p+2) \\ 1 & 4 \end{pmatrix}$

Worked example	Your turn
If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices, prove that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$	If $\mathbf{P}$ and $\mathbf{Q}$ are non-singular matrices, prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$

Let $C = (PQ)^{-1}$

$$(PQ)C = (PQ)(PQ)^{-1}$$
$$(PQ)C = I$$
$$P^{-1}PQC = P^{-1}I$$
$$IQC = P^{-1}$$
$$QC = P^{-1}$$
$$Q^{-1}QC = Q^{-1}P^{-1}$$
$$IC = Q^{-1}P^{-1}$$
$$C = Q^{-1}P^{-1}$$
$$(PQ)^{-1} = Q^{-1}P^{-1}$$

Worked example	Your turn
If A and B are non-singular matrices such that $\mathbf{ABA} = \mathbf{I}$, prove that $\mathbf{B} = \mathbf{A}^{-1}\mathbf{A}^{-1}$	If A and B are non-singular matrices such that $\mathbf{BAB} = \mathbf{I}$, prove that $\mathbf{A} = \mathbf{B}^{-1}\mathbf{B}^{-1}$

$$BAB = I$$
$$B^{-1}BAB = B^{-1}I$$
$$IAB = B^{-1}$$
$$AB = B^{-1}$$
$$ABB^{-1} = B^{-1}B^{-1}$$
$$AI = B^{-1}B^{-1}$$
$$A = B^{-1}B^{-1}$$

6.5) Inverting a 3 x 3 matrix

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Worked example	Your turn
If $\boldsymbol{A}=\begin{pmatrix}0 & -3 & -2\\ 1 & -4 & -2\\ -3 & 4 & 1\end{pmatrix}$, find $\boldsymbol{A}^{-1}$.	If $\boldsymbol{A}=\begin{pmatrix}1 & 3 & 1\\ 0 & 4 & 1\\ 2 & -1 & 0\end{pmatrix}$, find $\boldsymbol{A}^{-1}$. $\begin{pmatrix}-1 & 1 & 1\\ -2 & 2 & 1\\ 8 & -7 & -4\end{pmatrix}$

Worked example	Your turn
$\boldsymbol{A} = \begin{pmatrix} 5 & -4 & 4 \\ 8 & -7 & 8 \\ 2 & -2 & 3 \end{pmatrix}$, Show that $\mathbf{A}^{-1} = \mathbf{A}$.	$\boldsymbol{A} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$, Show that $\mathbf{A}^{-1} = \mathbf{A}$. $A^2 = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Worked example	Your turn
$\boldsymbol{A}=\begin{pmatrix}5 & -4 & 4\\ 8 & -7 & 8\\ 2 & -2 & 3\end{pmatrix}$, The matrix $\mathbf{B}$ is such that $(\mathbf{AB})^{-1}=\begin{pmatrix}2 & 5 & -3\\ -4 & 1 & -8\\ -1 & 0 & 11\end{pmatrix}$. Find $\boldsymbol{B}^{-1}$.	$\boldsymbol{A}=\begin{pmatrix}-2 & 3 & -3\\ 0 & 1 & 0\\ 1 & -1 & 2\end{pmatrix}$, The matrix $\mathbf{B}$ is such that $(\mathbf{AB})^{-1}=\begin{pmatrix}8 & -17 & 9\\ -5 & 10 & -6\\ -3 & 5 & -4\end{pmatrix}$. Find $\boldsymbol{B}^{-1}$.

$(AB)^{-1}=B^{-1}A^{-1}$

$(AB)^{-1}A=B^{-1}A^{-1}A$

$(AB)^{-1}A=B^{-1}$

$$B^{-1}=\begin{pmatrix}8 & -17 & 9\\ -5 & 10 & -6\\ -3 & 5 & -4\end{pmatrix}\begin{pmatrix}-2 & 3 & -3\\ 0 & 1 & 0\\ 1 & -1 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-7 & -2 & -6\\ 4 & 1 & 3\\ 2 & 0 & 1\end{pmatrix}$$

Worked example	Your turn
$\boldsymbol{A} = \begin{pmatrix} k & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix}, k \neq -1$	$\boldsymbol{A} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}, k \neq 1$
Find the inverse matrix of A in terms of k	Find the inverse matrix of A in terms of k
	$A^{-1} = \frac{1}{2-2k}\begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}$

Worked example	Your turn
Find the inverse of the matrix using elementary row operations $$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$	Find the inverse of the matrix using elementary row operations $$\begin{pmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{pmatrix}$$ $$\begin{pmatrix} -5 & 3 & -6 \\ -6 & 3 & -7 \\ -2 & 1 & -2 \end{pmatrix}$$

6.6) Solving systems of equations using matrices

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Worked example	Your turn
Solve the simultaneous equations: $6x - y + 2z = 6$ $-x + 2y - 6z = 3$ $2x - 3y - 5z = 24$	Solve the simultaneous equations: $-x + 6y - 2z = 21$ $6x - 2y - z = -16$ $-2x + 3y + 5z = 24$ $x = -1, y = 4, z = 2$

Worked example	Your turn
A llama farmer has three types of llama: woolly, classic and Suri. Initially his flock had 2810 llamas in it. There were 160 more woolly llamas than classic. After one year: • The number of woolly llamas had increased by 5% • The number of classic llamas had increased by 3% • The number of Suri llamas had decreased by 4% • Overall the flock size had increased by 46 Form and solve a matrix equation to find out how many of each type of llama there were in the initial flock.	A colony of 1000 mole-rats is made up of adult males, adult females and youngsters. Originally there were 100 more adult females than adult males. After one year: • The number of adult males had increased by 2% • The number of adult females had increased by 3% • The number of youngsters had decreased by 4% • The total number of mole-rats had decreased by 20 Form and solve a matrix equation to find out how many of each type of mole-rat were in the original colony. 100 adult males, 200 adult females, 700 youngsters in the original colony

Worked example	Your turn
The system of equations is consistent and has a single solution. Determine the possible values of k. $x - 3y - 2z = 7$ $kx - y + 3z = 11$ $x - y + z = 13$	The system of equations is consistent and has a single solution. Determine the possible values of k. $2x + 3y - z = 13$ $3x - y + kz = 11$ $x + y + z = 7$ $k \neq 15$

Worked example

A system of equations is shown below:

$$3x - ky - 6z = k$$
$$kx + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent.
If the system is consistent, determine whether there is a unique solution or an infinity of solutions.
In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 0$ (b) $k = -6$

Your turn

A system of equations is shown below:

$$3x - ky - 6z = k$$
$$kx + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent.
If the system is consistent, determine whether there is a unique solution or an infinity of solutions.
In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 1$

(a) $k = 1$: $\begin{vmatrix} 3 & -1 & -6 \\ 1 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = 0$

$3x - y - 6z = 1$ (1)
$x + 3y + 3z = 2$ (2)
$-3x - y + 3z = -2$ (3)
$(1) + 2 \times (2)$: $5x + 5y = 5$ (4)
$(2) - (3)$: $4x + 4y = 4$ (5)

Equations (4) and (5) are consistent so system is consistent and has an infinity of solutions. Planes meet at a sheaf

Worked example	Your turn

Worked example

A system of equations is shown below:

$$x - ry - 6z = r$$
$$rx - 4y - 12z = s$$
$$-3x + ty + 18z = u$$

For each of the following values of r, s and t, determine whether the system of equations is consistent or inconsistent.

If the system is consistent, determine whether there is a unique solution or an infinity of solutions.

In each case, identify the corresponding geometric configuration.

(a) $r = 2, s = 5, t = 4, u = 1$

(b) $r = 2, s = 4, t = 6, u = -6$

Your turn

A system of equations is shown below:

$$x - ry - 6z = r$$
$$rx - 4y - 12z = s$$
$$-3x + ty + 18z = u$$

For each of the following values of r, s and t, determine whether the system of equations is consistent or inconsistent.

If the system is consistent, determine whether there is a unique solution or an infinity of solutions.

In each case, identify the corresponding geometric configuration.

(a) $r = 2, s = 4, t = 6, u = -5$

(a) $\begin{vmatrix} 1 & -2 & -6 \\ 2 & -4 & -12 \\ -3 & 6 & 18 \end{vmatrix} = 0$

$$x - 2y - 6z = 2 \qquad (1)$$
$$2x - 4y - 12z = 4 \qquad (2)$$
$$-3x + 6y + 18z = -5 \qquad (3)$$

All three planes are parallel and non-identical.

The system of equations is inconsistent and has no solutions.