6) Hyperbolic functions

6.1) Introduction to hyperbolic functions

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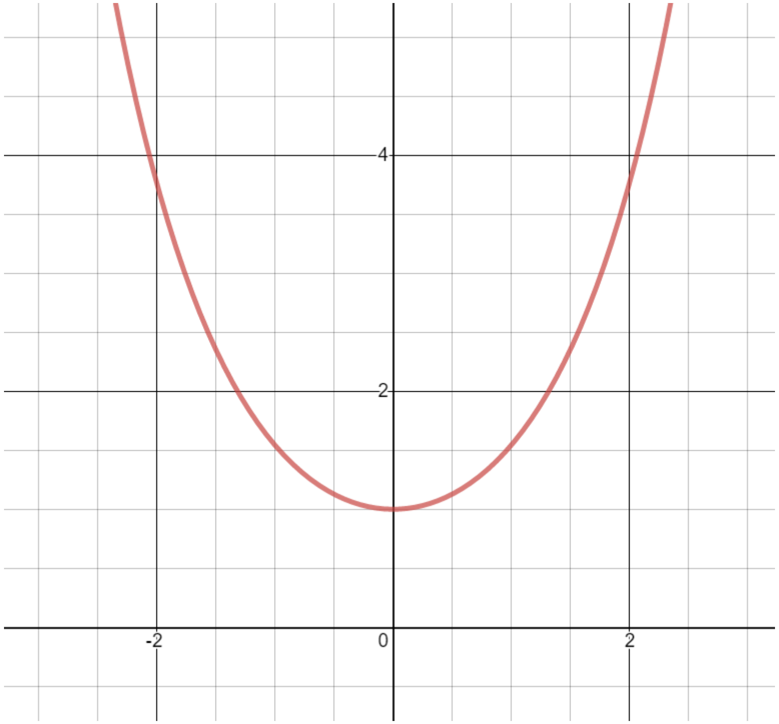
Worked example	Your turn
$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x = \frac{e^x + e^{-x}}{2}$
$cosech\, x =$	$sech\, x = \frac{2}{e^x + e^{-x}}$

Worked example	Your turn
Find to 2 decimal places, the values of:	Find to 2 decimal places, the values of:
$\sinh 2$	$\sinh 3$
	10.02
$\cosh 0$	$\cosh 1$
	1.54
$\tanh 1.8$	$\tanh 0.8$
	0.66

Worked example	Your turn
Find the exact values of:	Find the exact values of:
$\sinh(\ln 3)$	$\sinh(\ln 2)$
	$\frac{3}{4}$
$\cosh(\ln 2)$	$\cosh(\ln 3)$
	$\frac{5}{3}$
$\tanh(\ln 5)$	$\tanh(\ln 4)$
	$\frac{15}{17}$

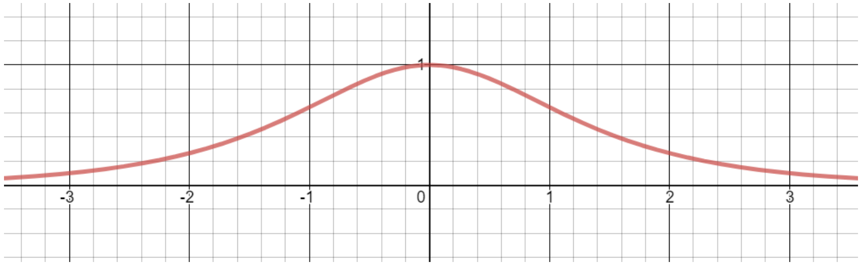
Worked example	Your turn
Find, to two decimal places, the value of x for which $$\cosh x = 3$$	Find, to two decimal places, the value of x for which $$\sinh x = 5$$ $$x = 2.31$$

Worked example	Your turn
Sketch the graph of $y = \sinh x$ by using the exponential definition and state the range	Sketch the graph of $y = \cosh x\,, x \in \mathbb{R}$ by using the exponential definition and state the range



$\cosh x \geq 1$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
By using the graph of $y = \sinh x$, sketch the graph of $y = cosech\ x$	By using the graph of $y = \cosh x$, Sketch the graph of $y = sech\ x$ 

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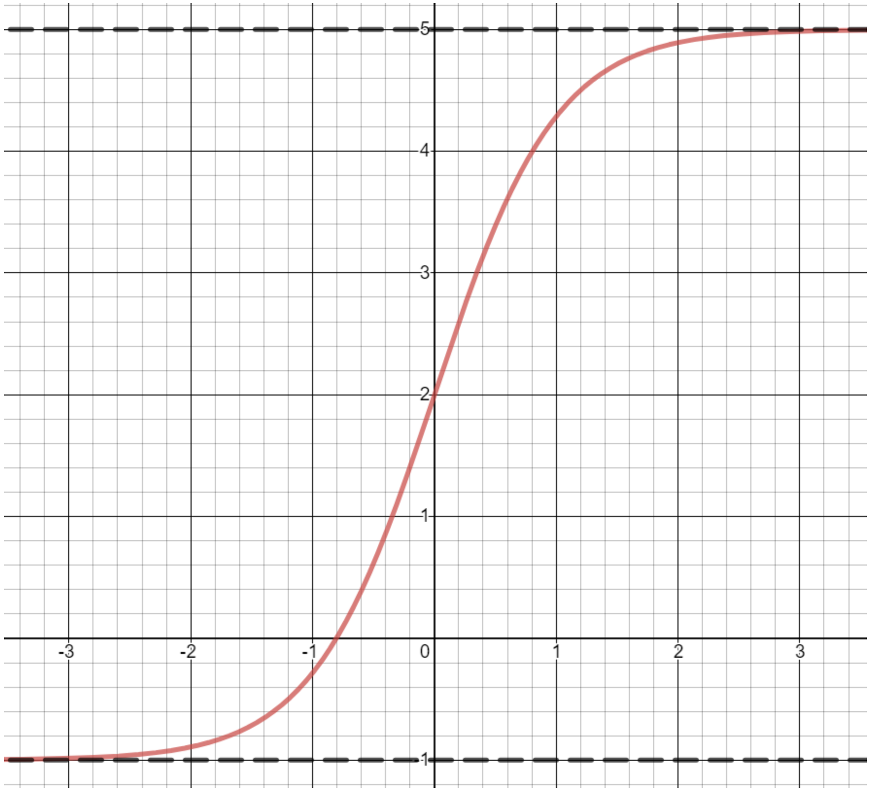
Worked example	Your turn
Sketch the graph of $y = 2\tanh x + 3$, stating the asymptote equations of the curve	Sketch the graph of $y = 3\tanh x + 2$, stating the asymptote equations of the curve



Asymptotes $y = 5$ and $y = -1$

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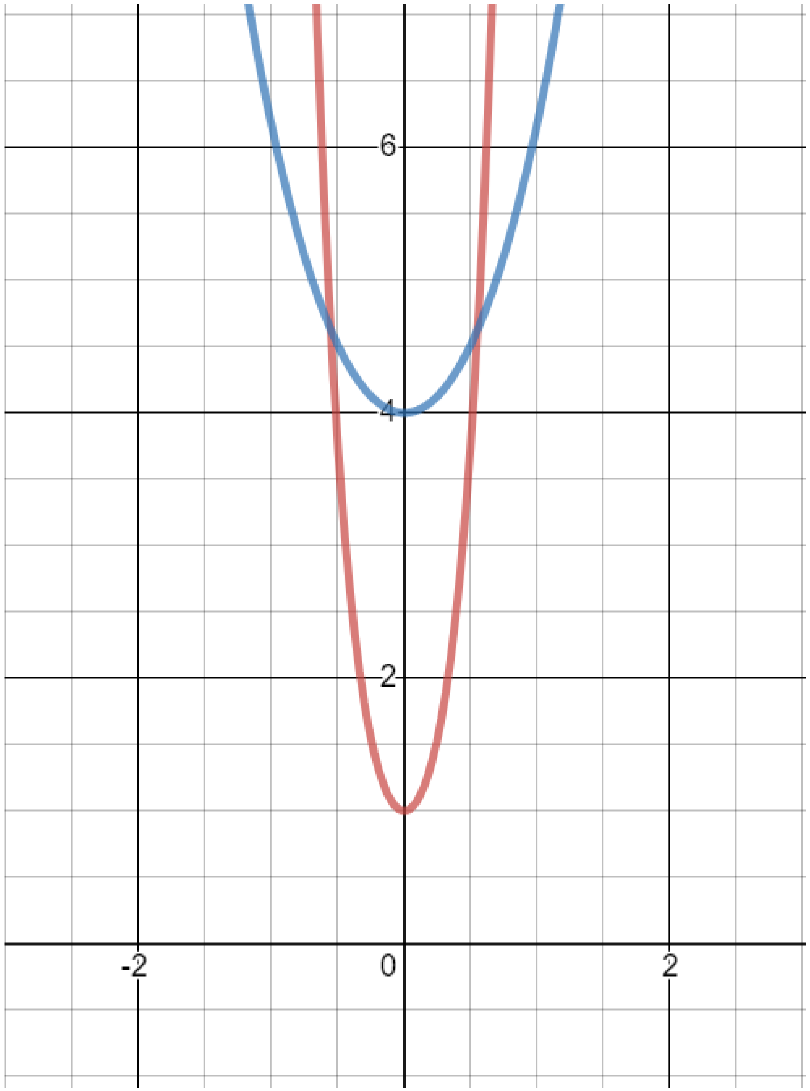
Worked example	Your turn
On the same diagram sketch the graphs of $y = \sinh 2x$ and $y = 2\sinh x$	On the same diagram sketch the graphs of $y = \cosh 4x$ and $y = 4\cosh x$



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6.2) Inverse hyperbolic functions

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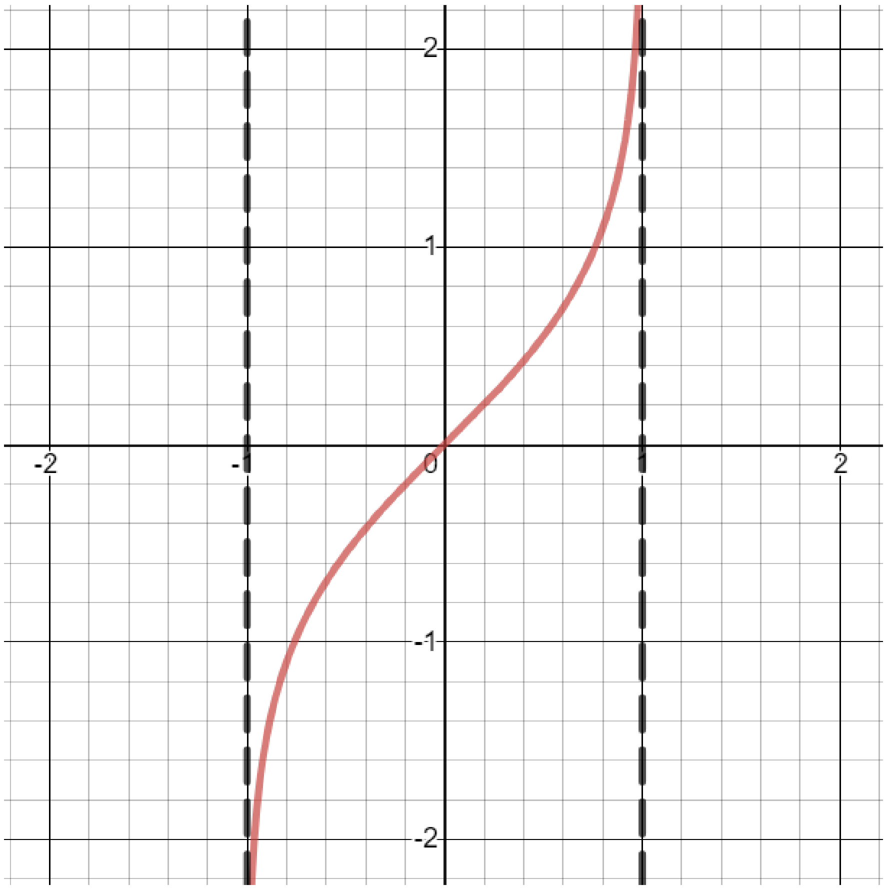
Worked example	Your turn
Sketch the graphs of:	Sketch the graphs of:

$$y = arsinh\, x$$

$$y = arcosh\, x$$

$$y = artanh\, x$$



Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Express as natural logarithms:	Express as natural logarithms:
$arsinh\ 2$	$arsinh\ 1$
	$\ln\left(1+\sqrt{2}\right)$
$arcosh\ 1$	$arcosh\ 2$
	$\ln(2+\sqrt{3})$
$artanh\ 3$	$artanh\ \frac{1}{3}$
	$\ln\sqrt{2}$

Worked example

Prove that $arcosh\ x = \ln\left(x + \sqrt{x^2 - 1}\right), x \geq 1$

Your turn

Prove that $arsinh\ x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$$y = arsinh\ x$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$e^y - e^{-y} = 2x$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= x \pm \sqrt{x^2 + 1}$$

Since $\sqrt{x^2 + 1} > x$, we can only use the positive case as $e^y > 0$

$$e^y = x + \sqrt{x^2 + 1}$$

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$arsinh\ x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Worked example	Your turn
Given that $artanh\ x + artanh\ y = ln\sqrt{5}$, find an expression for y in terms of x	Given that $artanh\ x + artanh\ y = ln\sqrt{3}$, prove that $y = \frac{2x-1}{x-2}$ Proof

6.3) Identities and equations

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Worked example	Your turn
Use definitions of $\sinh x$ and $\cosh x$ to prove that $sech^2 x = 1 - tanh^2 x$	Use definitions of $sinh\, x$ and $cosh\, x$ to prove that $\cosh^2 x - \sinh^2 x = 1$ Proof

Worked example

Use definitions of sinh x and cosh x to prove that:

$$sinh\ (A + B) = sinh\ A\ cosh\ B + cosh\ A\ sinh\ B$$

$$cosh\ (A - B) = cosh\ A\ cosh\ B - sinh\ A\ sinh\ B$$

Your turn

Use definitions of sinh x and cosh x to prove that:

$$sinh\ (A - B) = sinh\ A\ cosh\ B - cosh\ A\ sinh\ B$$

Proof

$$cosh\ (A + B) = cosh\ A\ cosh\ B + sinh\ A\ sinh\ B$$

Proof

Worked example	Your turn
Use definitions of $\sinh x$ and $\cosh x$ to prove that $\cosh 2x = 1 + 2\sinh^2 x$	Use definitions of $\sinh x$ and $\cosh x$ to prove that $\cosh 2x = 2\cosh^2 x - 1$ Proof

Worked example	Your turn
Use Osborn's rule to write down the hyperbolic identities corresponding to the trigonometric identities: $\cos 2x = \cos^4 x - \sin^4 x$	Use Osborn's rule to write down the hyperbolic identities corresponding to the trigonometric identities: $\cos 2x = \cos^2 x - \sin^2 x$ $\cosh 2x = \cosh^2 x + \sinh^2 x$

Worked example	Your turn
Given that $\sinh x = \frac{3}{5}$, find the exact value of:	Given that $\sinh x = \frac{3}{4}$, find the exact value of:
$\cosh x$	$\cosh x$ $\frac{5}{4}$
$\tanh x$	$\tanh x$ $\frac{3}{5}$
$\sinh 2x$	$\sinh 2x$ $\frac{15}{8}$

Worked example	Your turn
Solve for all real values of x: $6\sinh x + 2\cosh x = 7$	Solve for all real values of x: $6\sinh x - 2\cosh x = 7$ $x = \ln 4$

Worked example	Your turn
Solve for all real values of x: $$2\sinh^2 x - 5cosh\, x = 5$$	Solve for all real values of x: $$2\cosh^2 x - 5sinh\, x = 5$$ $$x = \ln\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$ $$x = \ln\left(3 + \sqrt{10}\right)$$

Worked example	Your turn
Solve for all real values of x: $$\cosh 2x - 5\sinh x + 2 = 0$$	Solve for all real values of x: $$\cosh 2x - 5\cosh x + 4 = 0$$ $$x = \ln\left(\frac{3\pm\sqrt{5}}{2}\right), x = 0$$

Worked example	Your turn
Solve the equation $$4\sinh 3x = 15 - 6e^{3x}$$ Give your answer in the form $\frac{1}{3}\ln k$, where k is an integer	Solve the equation $$3\sinh 2x = 13 - 3e^{2x}$$ Give your answer in the form $\frac{1}{2}\ln k$, where k is an integer $$x = \frac{1}{2}\ln 3$$

Worked example	Your turn
Express $5\cosh x + 3\sinh x$ in the form $R\cosh(x+\alpha)$, where $R > 0$. Give the value of α correct to 3 decimal places. Hence write down the minimum value of $10\cosh x + 6\sinh x$	Express $10\cosh x + 6\sinh x$ in the form $R\cosh(x+\alpha)$, where $R > 0$. Give the value of α correct to 3 decimal places. Hence write down the minimum value of $10\cosh x + 6\sinh x$ $8\cosh(x + 0.693)$ $\text{Minimum} = 8$

6.4) Differentiating hyperbolic functions

Worked example	Your turn
Prove that $\frac{d}{dx}(\sinh x) = \cosh x$	Prove that $\frac{d}{dx}(\cosh x) = \sinh x$ Proof

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$\sinh 7x$	$\sinh 2x$
	$2\cosh 2x$
$\cosh 6x$	$\cosh 3x$
	$3\sinh 3x$
$\tanh 5x$	$\tanh 4x$
	$4\operatorname{sech}^2 4x$

Worked example	Your turn
Differentiate with respect to x: $x^3 \sinh 5x$	Differentiate with respect to x: $x^2 \cosh 4x$ $2x \cosh 4x + 4x^2 \sinh 4x$

Worked example	Your turn
$y = \frac{1}{2}\ln(\tanh x)$ Show that $\frac{dy}{dx} = cosech\ 2x$	$y = \frac{1}{2}\ln(\coth x)$ Show that $\frac{dy}{dx} = -cosech\ 2x$ Shown

Worked example	Your turn
Prove that $$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$$	Prove that $$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$ $$y = arcosh\ x$$ $$x = \cosh y$$ $$\frac{dx}{dy} = \sinh y$$ $$\frac{dy}{dx} = \frac{1}{\sinh y}$$ $$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}}$$ $$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

Worked example	Your turn
Differentiate with respect to x:	Differentiate with respect to x:
$arsinh\ 7x$	$arsinh\ 2x$
	$\dfrac{2}{\sqrt{4x^2+1}}$
$arcosh\ 6x$	$arcosh\ 3x$
	$\dfrac{3}{\sqrt{9x^2-1}}$
$artanh\ 5x$	$artanh\ 4x$
	$\dfrac{4}{1-16x^2}$

Worked example	Your turn
Given that $y = (arsinh\ x)^4$ prove that $(x^2 + 1)\left(\frac{dy}{dx}\right)^2 = 16y^{\frac{3}{2}}$	Given that $y = (arcosh\ x)^2$ prove that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y$ Proof

Worked example	Your turn
(a) Show that $\frac{d}{dx}(arsinh\ x) = \frac{1}{\sqrt{1+x^2}}$ (b) Find the first two non-zero terms of the series expansion of $arsinh\ x$. The general form for the series expansion of $arsinh\ x$ is given by $$arsinh\ x = \sum_{r=0}^{\infty}\left(\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{2n+1}}{2n+1}$$ (c) Find, in simplest terms, the coefficient of x^7. (d) Use your approximation up to and including the term in x^7 to find an approximate value for $arsinh\ 0.5$. (e) Calculate the percentage error in using this approximation.	(a) Show that $\frac{d}{dx}(arsinh\ x) = \frac{1}{\sqrt{1+x^2}}$ (b) Find the first two non-zero terms of the series expansion of $arsinh\ x$. The general form for the series expansion of $arsinh\ x$ is given by $$arsinh\ x = \sum_{r=0}^{\infty}\left(\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{2n+1}}{2n+1}$$ (c) Find, in simplest terms, the coefficient of x^5. (d) Use your approximation up to and including the term in x^5 to find an approximate value for $arsinh\ 0.5$. (e) Calculate the percentage error in using this approximation. (a) Shown (b) $x - \frac{1}{6}x^3$ (c) $\frac{3}{40}$ (d) $0.48151\ldots$ (e) 0.062% (3 d.p.)

Worked example	Your turn
Find the exact coordinates of the stationary point on the curve with equation $y = 6\cosh x - \sinh x$	Find the exact coordinates of the stationary point on the curve with equation $y = 12\cosh x - \sinh x$ $\left(\frac{1}{2}\ln\frac{13}{11}, \sqrt{143}\right)$

Worked example

Find the first three non-zero terms of the Maclaurin series for $\sinh x$
Hence find the percentage error when this approximation is used to evaluate $\sinh 0.4$

Your turn

Find the first three non-zero terms of the Maclaurin series for $\cosh x$
Hence find the percentage error when this approximation is used to evaluate $\cosh 0.2$

$$1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$0.0000087\%$$

6.5) Integrating hyperbolic functions

Worked example	Your turn
Find:	Find:
$\int \cosh(3x-2)\,dx$	$\int \cosh(4x-1)\,dx$ $\frac{1}{4}\sinh(4x-1)+c$
$\int sinh\left(\frac{5}{7}x\right)\,dx$	$\int sinh\left(\frac{2}{3}x\right)\,dx$ $\frac{3}{2}\cosh\left(\frac{2}{3}x\right)+c$
$\int \frac{7}{\sqrt{1+x^2}}dx$	$\int \frac{3}{\sqrt{1+x^2}}dx$ $3\,arsinh\,x+c$
$\int \frac{6}{\sqrt{x^2-1}}dx$	$\int \frac{4}{\sqrt{x^2-1}}dx$ $4\,arcosh\,x+c$
$\int sinh\,(5x)\,\,dx$	$\int sinh\,(3x)\,\,dx$ $\frac{1}{3}\cosh(3x)+c$
$\int \frac{4}{\sqrt{x^2-1}}dx$	$\int \frac{10}{\sqrt{x^2-1}}dx$ $10\,arcosh\,x+c$
$\int \frac{3}{\sqrt{1+x^2}}dx$	$\int \frac{2}{\sqrt{1+x^2}}dx$ $2\,arsinh\,x+c$

Worked example	Your turn
Find: $$\int \frac{3-7x}{\sqrt{x^2+1}}\,dx$$	Find: $$\int \frac{2+5x}{\sqrt{x^2+1}}\,dx$$ $2\,arsinh\,x + 5\sqrt{1+x^2} + c$

Worked example	Your turn
Find: $$\int \sinh^7 3x \cosh 3x \; dx$$	Find: $$\int \cosh^5 2x \sinh 2x \; dx$$ $$\frac{1}{12}\cosh^6 2x + c$$

Worked example	Your turn
Find: $$\int \coth x \; dx$$	Find: $$\int \tanh x \; dx$$ $$ln\lvert cosh\, x\rvert + C$$

Worked example	Your turn
Find: $$\int \sinh^2 5x \;\; dx$$	Find: $$\int \cosh^2 3x \;\; dx$$ $$\frac{1}{2}x + \frac{1}{12}\sinh 6x + c$$

Worked example	Your turn
Find: $$\int \cosh^3 x \ dx$$	Find: $$\int \sinh^3 x \ dx$$ $$\frac{1}{3}\cosh^3 x - \cosh x + c$$

Worked example	Your turn
Find: $$\int e^{3x} \cosh x \; dx$$	Find: $$\int e^{2x} \sinh x \; dx$$ $$\frac{1}{6}(e^{3x} - 3e^{x}) + c$$

Worked example	Your turn
Find: $$\int cosech\, x\, dx$$ (requires partial fractions)	Find: $$\int \operatorname{sech} x\ \ dx$$ $2\arctan(e^{x}) + c$

Worked example	Your turn
Show that $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = arsinh\left(\frac{x}{a}\right) + c$	Show that $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = arcosh\left(\frac{x}{a}\right) + c$ Shown

Worked example	Your turn
Evaluate $\int_5^8 \frac{1}{\sqrt{x^2+16}}\,dx$	Evaluate $\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx$ $\ln\left(\frac{2+\sqrt{3}}{2}\right)$

Worked example	Your turn
By using a hyperbolic substitution, show that $$\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}arcosh\,x + c$$	By using a hyperbolic substitution, show that $$\int \sqrt{1+x^2}\,dx = \frac{1}{2}arsinh\,x + \frac{1}{2}x\sqrt{1+x^2} + c$$ Shown using $x = \sinh u$ NB: Can also use $x = \tan u$ but longer

Worked example	Your turn
Using a hyperbolic substitution, evaluate $$\int_4^8 \frac{x^3}{\sqrt{x^2-16}}\,dx$$	Using a hyperbolic substitution, evaluate $$\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\,dx$$ $$18\sqrt{5}+18$$

Worked example	Your turn
Find: $$\int \frac{1}{\sqrt{12x+3x^2}}\,dx$$	Find: $$\int \frac{1}{\sqrt{12x+2x^2}}\,dx$$ $$\frac{1}{\sqrt{2}} arcosh\left(\frac{x+3}{3}\right)+C$$

Worked example	Your turn
Find: $$\int \frac{1}{x^2 - 6x - 3}\,dx$$	Find: $$\int \frac{1}{x^2 - 8x + 8}\,dx$$ $$\frac{\sqrt{2}}{8}\ln\left|\frac{x - 4 - 2\sqrt{2}}{x - 4 + 2\sqrt{2}}\right| + C$$

Worked example	Your turn
Evaluate: $$\int_0^1 \frac{1}{\sqrt{x^2+8x+17}}\,dx$$	Evaluate: $$\int_0^1 \frac{1}{\sqrt{x^2+2x+5}}\,dx$$ 0.400 (3 sf)

Worked example	Your turn
Find: $$\int \frac{1}{\sqrt{9x^2 + 4}}\, dx$$	Find: $$\int \frac{1}{\sqrt{4x^2 + 9}}\, dx$$ $$\frac{1}{2} arsinh \left(\frac{2x}{3}\right) + c$$

Worked example	Your turn
$$25x^2 + 10x + 17 \equiv a(x+b)^2 + c$$ a) Find the values of a, b and c Hence, or otherwise, find: b) $\int \frac{1}{25x^2+10x+17}\,dx$ c) $\int \frac{1}{\sqrt{25x^2+10x+17}}\,dx$	$$9x^2 + 6x + 5 \equiv a(x+b)^2 + c$$ a) Find the values of a, b and c Hence, or otherwise, find: b) $\int \frac{1}{9x^2+6x+5}\,dx$ c) $\int \frac{1}{\sqrt{9x^2+6x+5}}\,dx$ a) $a = 9, b = \frac{1}{3}, c = 4$ b) $\frac{1}{6}\arctan\frac{3x+1}{2} + c$ c) $\frac{1}{3} arsinh\frac{3x+1}{2} + c$