5) Polar coordinates

5.1) Polar coordinates and equations

Chapter CONTENTS

Worked example	Your turn
Convert from Cartesian to polar coordinates: $(4, -3)$	Convert from Cartesian to polar coordinates: $(3, 4)$ $(5, 0.927)$
$(-5, 12)$	$(5, -12)$ $(13, -1.176)$
$(\sqrt{3}, 1)$	$(-\sqrt{3}, -1)$ $\left(2, \frac{7\pi}{6}\right)$ or $\left(2, -\frac{5\pi}{6}\right)$

Worked example

Convert from polar to Cartesian coordinates:

$$\left(8, \frac{-5\pi}{3}\right)$$

$$\left(4, \frac{\pi}{3}\right)$$

$$\left(3, \frac{\pi}{2}\right)$$

Your turn

Convert from polar to Cartesian coordinates:

$$\left(10, \frac{4\pi}{3}\right)$$

$$\left(-5, 5\sqrt{3}\right)$$

$$\left(8, \frac{2\pi}{3}\right)$$

$$\left(-4, 4\sqrt{3}\right)$$

$$(2, \pi)$$

$$(-2, 0)$$

Worked example

Find Cartesian equations for the following curves:

$$r = 4$$

$$r = 3 + \cos 4\theta$$

$$r^4 = \sin 2\theta, \qquad 0 < \theta \leq \frac{\pi}{2}$$

Your turn

Find Cartesian equations for the following curves:

$$r = 5$$

$$x^2 + y^2 = 25$$

$$r = 2 + \cos 2\theta$$

$$(x^2 + y^2)^{\frac{3}{2}} = 3x^2 + y^2$$

$$r^2 = \sin 2\theta, \qquad 0 < \theta \leq \frac{\pi}{2}$$

$$(x^2 + y^2)^2 = 2xy$$

Worked example	Your turn
Find Cartesian equations for the following curves:	Find Cartesian equations for the following curves:
$r = 5\sec\theta$	$r = 3\sec\theta$
	$x = 3$
$r = 3cosec\ \theta$	$r = 5cosec\ \theta$
	$y = 5$
$r = 4\cos\theta$	$r = 2\cos\theta$
	$x^2 + y^2 = 2x$ or $(x - 1)^2 + y^2 = 1$
$r = 2\sin\theta$	$r = 4\sin\theta$
	$x^2 + y^2 = 4y$ or $x^2 + (y - 2)^2 = 4$

Worked example	Your turn
Find Cartesian equations for the following curves: $r = 8\cot\theta \operatorname{cosec}\theta$	Find Cartesian equations for the following curves: $r = 4\tan\theta\sec\theta$
	$x^2 = 4y$ or $y = \frac{x^2}{4}$
$r^2 = 1 + \cot^2\theta$	$r^2 = 1 + \tan^2\theta$
	$x^2 = 1$ or $x = \pm 1$

Worked example	Your turn
Find polar equations for the following curves:	Find polar equations for the following curves:
$y^2 = 2x$	$y^2 = 4x$
	$r = 4\cot\theta\ cosec\ \theta$
$x^2 - y^2 = 10$	$x^2 - y^2 = 5$
	$r^2 = 5\sec 2\theta$
$y\sqrt{2} = x + 8$	$y\sqrt{3} = x + 4$
	$r = 2cosec\left(\theta - \frac{\pi}{6}\right)$

Worked example	Your turn
Find polar equations for the following curves:	Find polar equations for the following curves:
$y = 4x$	$y = 2x$
	$\tan\theta = 2$
$xy = 8$	$xy = 4$
	$r^2 = 8\ cosec\ 2\theta$
$y = -\sqrt{2}x + 4$	$y = -\sqrt{3}x + 4$
	$r = 2\ cosec\left(\theta + \frac{\pi}{3}\right)$

Worked example	Your turn
Find polar equations for the following curves:	Find polar equations for the following curves:
$x^2 + y^2 - 4x = 0$	$x^2 + y^2 - 2x = 0$
	$r = 2\cos\theta$
$(x+y)^2 = 8$	$(x+y)^2 = 4$
	$r^2 = \dfrac{4}{1+\sin 2\theta}$
$x - y = 5$	$x - y = 3$
	$r = \dfrac{3}{\sqrt{2}}\sec\left(\theta + \dfrac{\pi}{4}\right)$

5.2) Sketching curves

Worked example

Sketch the following curves:

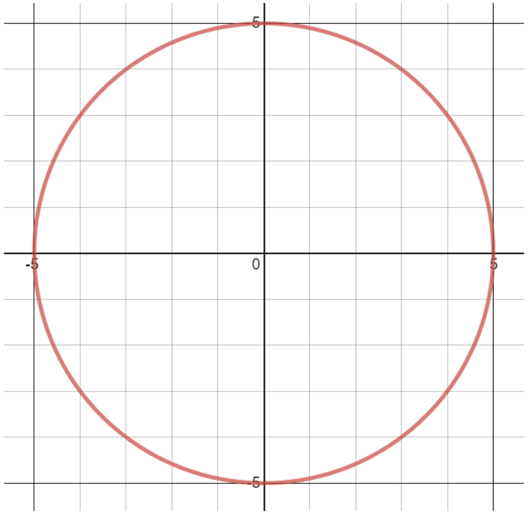
$$r = 4$$

$$r = 6$$

Your turn

Sketch the following curve:

$$r = 5$$



$$r = a$$

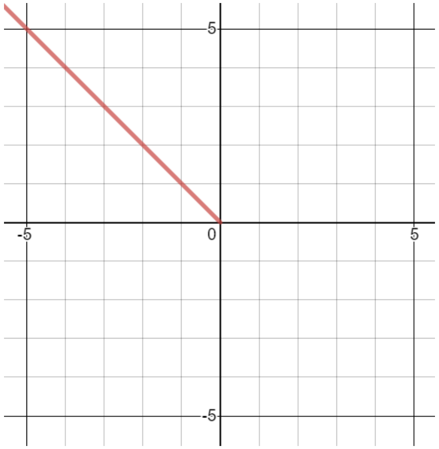
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$\theta = -\frac{3\pi}{4}$$

$$\theta = \frac{\pi}{4}$$

$$\theta = \frac{3\pi}{4}$$



$$\theta = a$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

Worked example

Sketch the following curves:

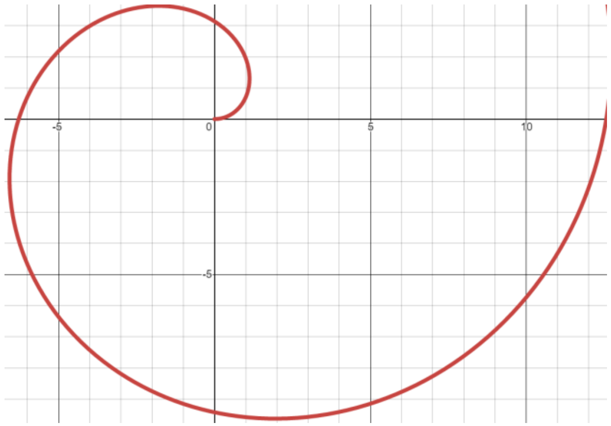
$$r = \theta$$

$$r = 3\theta$$

Your turn

Sketch the following curves:

$$r = 2\theta$$



$$r = a\theta$$

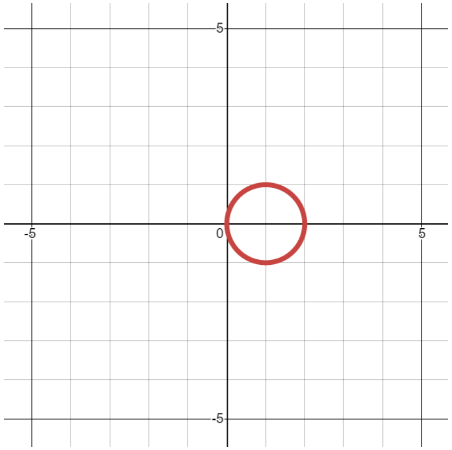
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r = \cos\theta$$

$$r = 3\cos\theta$$

$$r = 2\cos\theta$$



$$r = a\cos\theta$$

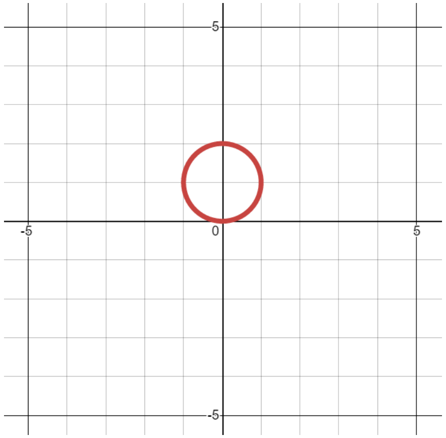
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r = \sin\theta$$

$$r = 3\sin\theta$$

$$r = 2\sin\theta$$



$$r = a\sin\theta$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example

Sketch the following curves:

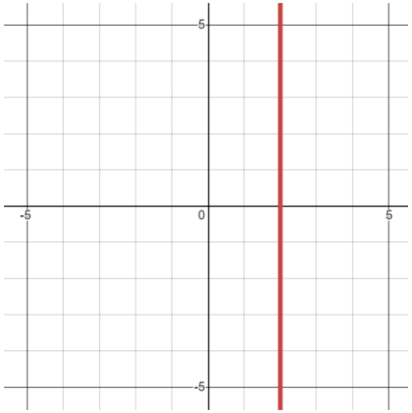
$$r = \sec\theta$$

$$r = 3\sec\theta$$

Your turn

Sketch the following curves:

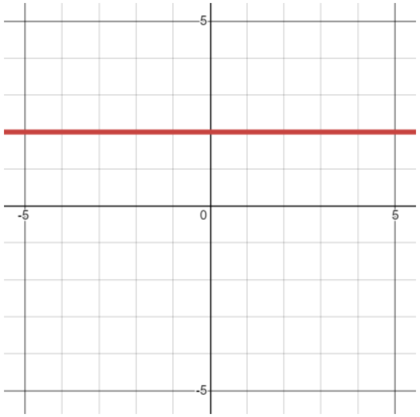
$$r = 2\sec\theta$$



$$r = a\sec\theta$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves: $r = cosec\, \theta$	Sketch the following curves: $r = 2\ \text{cosec}\, \theta$



$$r = 3\, cosec\, \theta$$

$$r = a\ \text{cosec}\, \theta$$

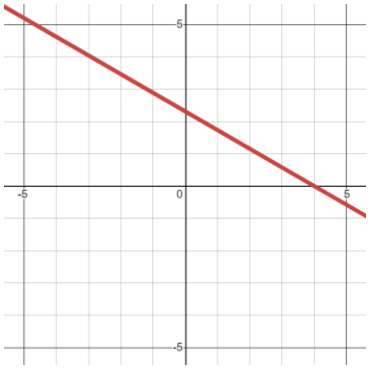
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r = \sec\left(\theta - \frac{\pi}{4}\right)$$

$$r = 3\sec\left(\theta + \frac{\pi}{6}\right)$$

$$r = 2\sec\left(\theta - \frac{\pi}{3}\right)$$



$$r = a\sec(\theta - k)$$

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Worked example

Sketch the following curves:

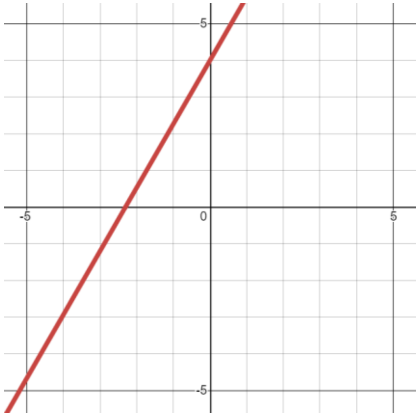
$$r = \operatorname{cosec}\left(\theta - \frac{\pi}{4}\right)$$

$$r = 3\operatorname{cosec}\left(\theta + \frac{\pi}{6}\right)$$

Your turn

Sketch the following curves:

$$r = 2\operatorname{cosec}\left(\theta - \frac{\pi}{3}\right)$$



$$r = a\operatorname{cosec}(\theta - k)$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example

Sketch the following curves:

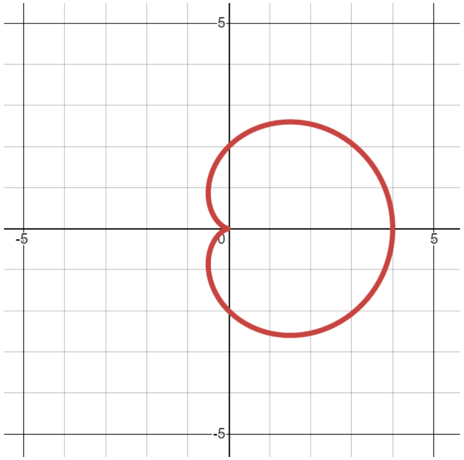
$$r = 1 + \cos\theta$$

$$r = 3(1 + \cos\theta)$$

Your turn

Sketch the following curves:

$$r = 2(1 + \cos\theta)$$



$$r = a(1 + \cos\theta)$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

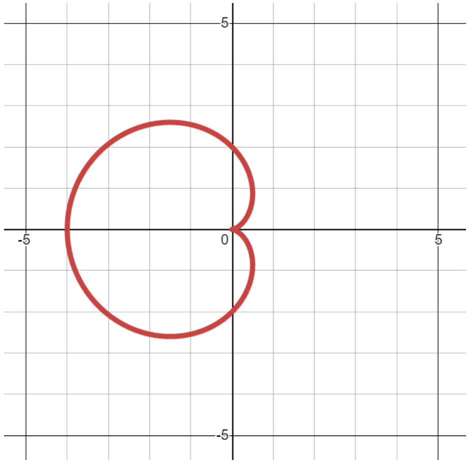
Worked example

$$r = 1 - \cos\theta$$

$$r = 3(1 - \cos\theta)$$

Your turn

$$r = 2(1 - \cos\theta)$$



$$r = a(1 - \cos\theta)$$

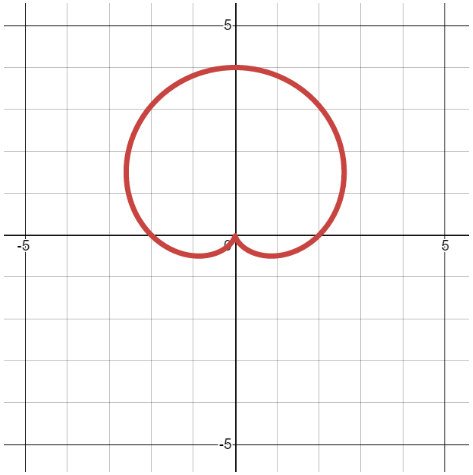
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r = 1 + \sin\theta$$

$$r = 3(1 + \sin\theta)$$

$$r = 2(1 + \sin\theta)$$



$$r = a(1 + \sin\theta)$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

Worked example

Sketch the following curves:

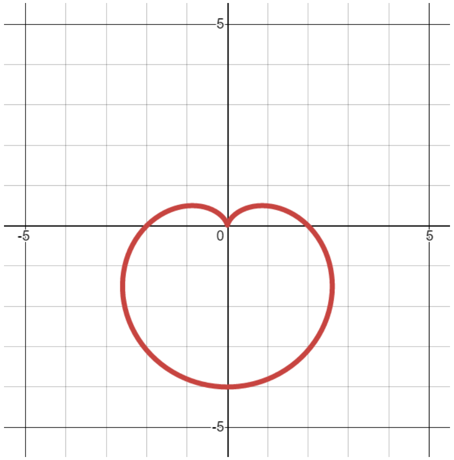
$$r = 1 - \sin\theta$$

$$r = 3(1 - \sin\theta)$$

Your turn

Sketch the following curves:

$$r = 2(1 - \sin\theta)$$



$$r = a(1 - \sin\theta)$$

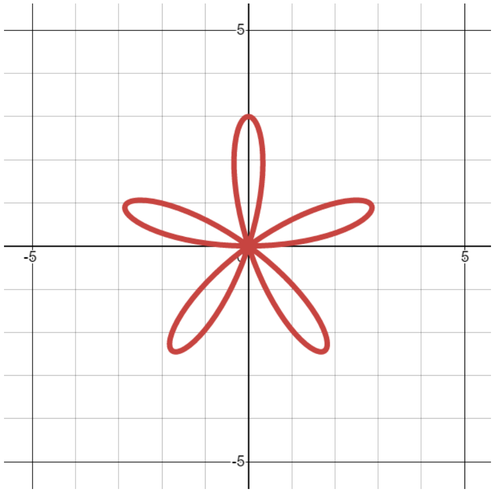
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r = 2\sin 3\theta$$

$$r = 5\sin 7\theta$$

$$r = 3\sin 5\theta$$



$$r = a\sin n\theta$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn

Worked example

Sketch the following curves:

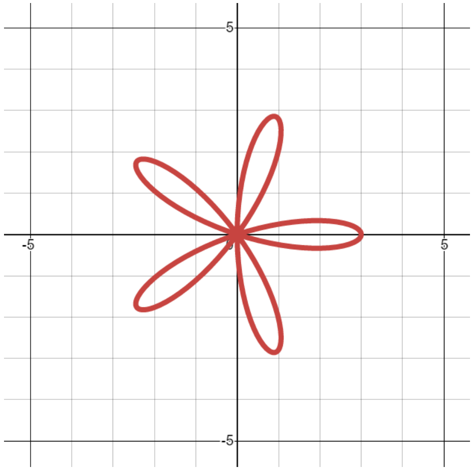
$$r = 2\cos 3\theta$$

$$r = 5\cos 7\theta$$

Your turn

Sketch the following curves:

$$r = 3\cos 5\theta$$



$$r = a\cos n\theta$$

Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves: $r^2 = 16\cos 2\theta$	Sketch the following curves: $r^2 = 4\cos 2\theta$

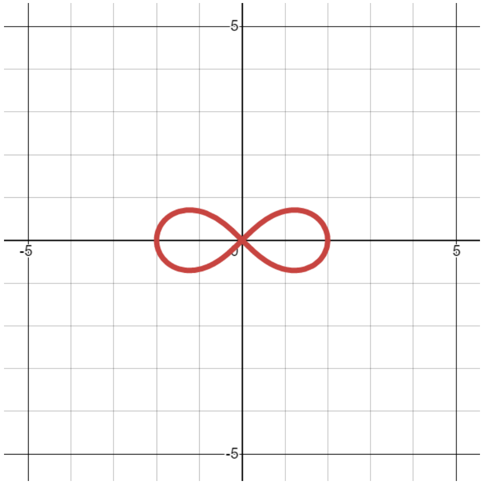
Worked example:

$$r^2 = 16\cos 2\theta$$

$$r^2 = 9\cos 2\theta$$

Your turn:

$$r^2 = 4\cos 2\theta$$



$$r^2 = a^2\cos 2\theta$$

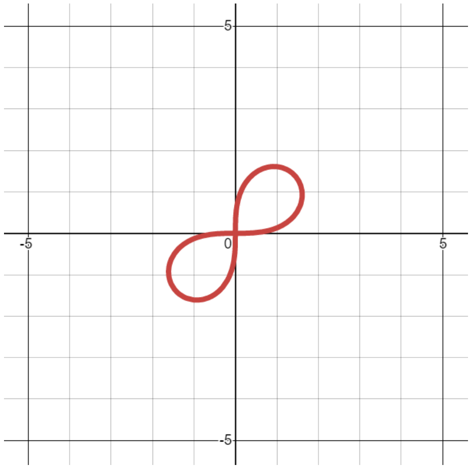
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r^2 = 16\sin 2\theta$$

$$r^2 = 9\sin 2\theta$$

$$r^2 = 4\sin 2\theta$$



$$r^2 = a^2\sin 2\theta$$

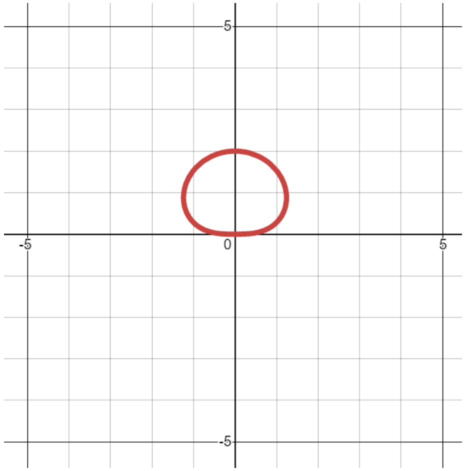
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r^2 = 16\cos\theta$$

$$r^2 = 9\cos\theta$$

$$r^2 = 4\cos\theta$$



$$r^2 = a^2\cos\theta$$

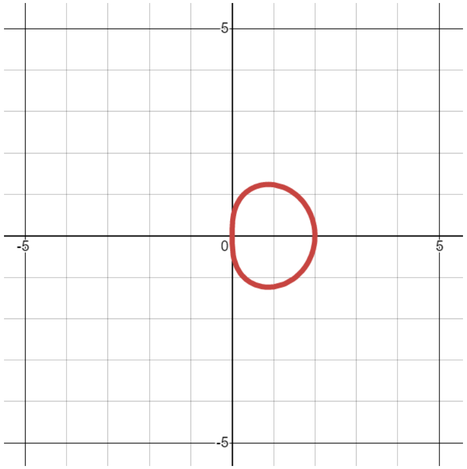
Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch the following curves:	Sketch the following curves:

$$r^2 = 16\sin\theta$$

$$r^2 = 9\sin\theta$$

$$r^2 = 4\sin\theta$$



$$r^2 = a^2\sin 2\theta$$

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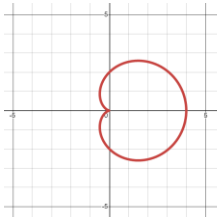
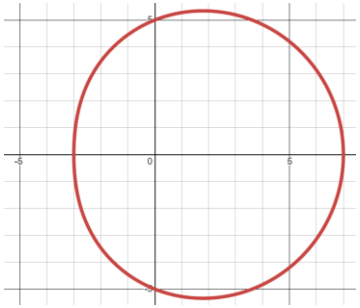
Worked example	Your turn
Sketch: $r = 6(3 + 3\cos\theta)$	Sketch: $r = a(2 + 2\cos\theta)$

Sketches with intercepts and axes labelled. General shapes:



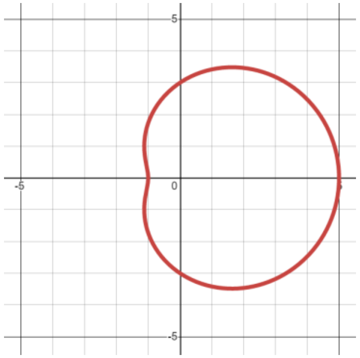
$r = 5(3 + 2\cos\theta)$

$r = a(5 + 2\cos\theta)$



$r = 4(3 + \cos\theta)$

$r = a(3 + 2\cos\theta)$



Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Sketch: $r = 6(3 + 3\sin\theta)$	Sketch: $r = a(2 + 2\sin\theta)$

Sketch:

$$r = 6(3 + 3\sin\theta)$$

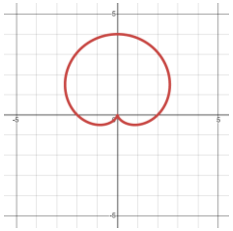
$$r = 5(3 + 2\sin\theta)$$

$$r = 4(3 + \sin\theta)$$

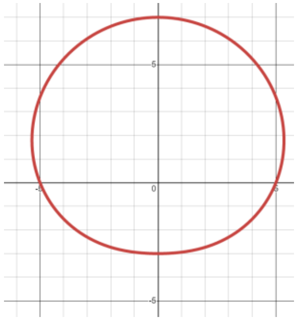
Sketch:

$$r = a(2 + 2\sin\theta)$$

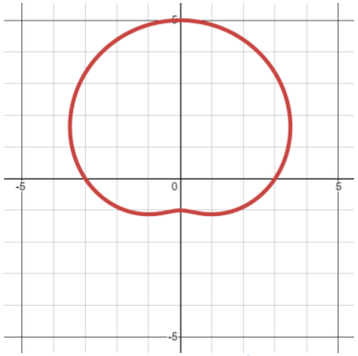
Sketches with intercepts and axes labelled. General shapes:



$$r = a(5 + 2\sin\theta)$$



$$r = a(3 + 2\sin\theta)$$



Graphs used with permission from DESMOS: https://www.desmos.com/

Worked example	Your turn
Show on an Argand diagram the locus of points given by the values of z satisfying $\lvert z+4+3i\rvert = 5$ Show that this locus of points can be represented by the polar curve $r = -8\cos\theta - 6\sin\theta$	Show on an Argand diagram the locus of points given by the values of z satisfying $\lvert z-3-4i\rvert = 5$ Show that this locus of points can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$ Shown

5.3) Area enclosed by a polar curve

Chapter CONTENTS

Worked example	Your turn
Find the area enclosed by the cardioid with equation $r = a(1 + \sin\theta)$	Find the area enclosed by the cardioid with equation $r = a(1 + \cos\theta)$ $\frac{3a^2\pi}{2}$

Worked example	Your turn
Find the area of one loop of the curve with polar equation $y = a\cos 3\theta$	Find the area of one loop of the curve with polar equation $y = a\sin 4\theta$ $\frac{a^2\pi}{16}$

Worked example	Your turn
A curve has equation $r = a + 3\cos\theta\,,\ a > 0$ The area enclosed by the curve is $\frac{107}{2}\pi$. Find the value of a.	A curve has equation $r = a + 5\sin\theta\,,\ a > 5$ The area enclosed by the curve is $\frac{187}{2}\pi$. Find the value of a. $a = 9$

Worked example	Your turn
Find the exact value of the area of the finite region contained within both curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$	Find the exact value of the area of the finite region contained within both two curves $r = 2 + \cos\theta$ and $r = 5\cos\theta$ $\frac{43\pi}{12} - \sqrt{3}$

Worked example	Your turn
The set of points, A, is defined by: $A = \left\{z: \frac{\pi}{2} \leq \arg z \leq \pi\right\} \cap \{z: \lvert z+12-5i\rvert \leq 13$ Find the area of the region defined by A	The set of points, A, is defined by: $A = \left\{z: -\frac{\pi}{4} \leq \arg z \leq 0\right\} \cap \{z: \lvert z-4+3i\rvert \leq 5$ Find the area of the region defined by A 35.1 (3 sf)

Worked example	Your turn
Two curves are given by the polar equations $r = 3, \qquad 0 \le \theta < \frac{\pi}{2}$ $r = 2.5 + \sin 5\theta \qquad 0 \le \theta \le \frac{2\pi}{5}$ Find the area of the region enclosed between the two curves where $r > 2$ and $r < 1.5 + \sin 3\theta$	Two curves are given by the polar equations $r = 2, \qquad 0 \le \theta < \frac{\pi}{2}$ $r = 1.5 + \sin 3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$ Find the area of the region enclosed between the two curves where $r > 2$ and $r < 1.5 + \sin 3\theta$ $\frac{13\sqrt{3}}{24} - \frac{5\pi}{36}$

5.4) Tangents to polar curves

Chapter CONTENTS

Worked example	Your turn
Find the coordinates of the points on $r = a(1 + \sin\theta)$ where the tangents are parallel to the initial line $\theta = 0$.	Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are parallel to the initial line $\theta = 0$. $\left(\frac{3a}{2}, \pm\frac{\pi}{3}\right)$ and $(0, \pi)$

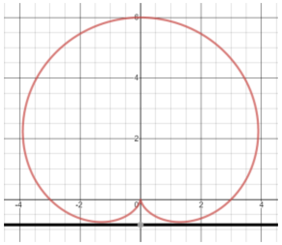
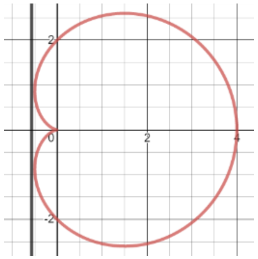
Worked example	Your turn
Find the coordinates of the points on $r = a(1 + \sin\theta)$ where the tangents are perpendicular to the initial line $\theta = 0$.	Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are perpendicular to the initial line $\theta = 0$. $(2a, 0), (0, \pi)$ and $\left(\frac{(2-\sqrt{2})}{2}a, \pm\frac{3\pi}{4}\right)$

Worked example	Your turn
The curve C has polar equation $$r = 1 + 3\cos\theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}$$ At the point P on C, the tangent to C is parallel to the initial line. Given that O is the pole, find the exact length of the line OP.	The curve C has polar equation $$r = 1 + 2\cos\theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}$$ At the point P on C, the tangent to C is parallel to the initial line. Given that O is the pole, find the exact length of the line OP. $$\frac{3+\sqrt{33}}{4}$$

Worked example	Your turn
Find the points on the spiral $r = e^{3\theta}, 0 \le \theta \le \pi$, where the tangents are: • Perpendicular to the initial line • Perpendicular to the initial line	Find the points on the spiral $r = e^{2\theta}, 0 \le \theta \le \pi$, where the tangents are: • Perpendicular to the initial line $(9.15, 1.11)$ • Perpendicular to the initial line $(212, 2.68)$

Worked example	Your turn
Find the equation and the points of contact of the tangents to the curve $$r = a\cos 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$$ that are parallel to the initial line	Find the equation and the points of contact of the tangents to the curve $$r = a\sin 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$$ that are parallel to the initial line $(0, 0)$; Tangent $\theta = 0$ $\left(\frac{2a\sqrt{2}}{3}, 0.955\right)$; Tangent $r = \frac{4a}{3\sqrt{3}}\ cosec\ \theta$

Worked example	Your turn
Find the equation and the points of contact of the tangents to the curve $$r = a\cos 2\theta, 0 \le \theta \le \frac{\pi}{2}$$ that are perpendicular to the initial line.	Find the equation and the points of contact of the tangents to the curve $$r = a\sin 2\theta, 0 \le \theta \le \frac{\pi}{2}$$ that are perpendicular to the initial line. $(0, 0)$; Tangent $\theta = \frac{\pi}{2}$ $\left(\frac{2a\sqrt{2}}{3}, 0.615\right)$; Tangent $r = \frac{4a}{3\sqrt{3}}\ sec\ \theta$

Worked example	Your turn
The diagram shows the cardioid with polar equation $r = 4(1 + \sin\theta)$ An area is enclosed by the curve and the horizontal line segment which is tangent to the curve and parallel to the initial line. Find the area. 	The diagram shows the cardioid with polar equation $r = 2(1 + \cos\theta)$ An area is enclosed by the curve and the vertical line segment which is tangent to the curve and perpendicular to the initial line. Find the area.  $\frac{15\sqrt{3}}{4} - 2\pi$

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