4.4) Expressions relating to the roots of a polynomial

Worked example

A quadratic equation has roots α and β. Given that $\alpha + \beta = 3$ and $\alpha\beta = 4$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ (b) $\alpha^2\beta^2$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$

Your turn

A quadratic equation has roots α and β. Given that $\alpha + \beta = 4$ and $\alpha\beta = 3$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ (b) $\alpha^2\beta^2$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$

(a) $\frac{4}{3}$
(b) 9
(c) 10
(d) 28

Worked example

A quadratic equation has roots α and β. Given that $\alpha + \beta = 3$ and $\alpha\beta = 4$, find:

(a) $(\alpha + 3)(\beta + 3)$

(b) $(\alpha^2 - 5)(\beta^2 - 5)$

Your turn

A quadratic equation has roots α and β. Given that $\alpha + \beta = 4$ and $\alpha\beta = 3$, find:

(a) $(\alpha + 5)(\beta + 5)$

(b) $(\alpha^2 - 3)(\beta^2 - 3)$

(a) 48

(b) -12

Worked example

A cubic equation has roots α, β and γ.
Given that $\alpha + \beta + \gamma = -2$, $\alpha\beta + \alpha\gamma + \beta\gamma = 3$ and $\alpha\beta\gamma = -4$ find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
(b) $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$
(d) $(\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2$
(e) $\alpha^2\beta^2\gamma^2$

Your turn

A cubic equation has roots α, β and γ.
Given that $\alpha + \beta + \gamma = 2$, $\alpha\beta + \alpha\gamma + \beta\gamma = -3$ and $\alpha\beta\gamma = 4$ find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
(b) $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$
(d) $(\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2$
(e) $\alpha^3\beta^3\gamma^3$

(a) $-\frac{3}{4}$
(b) 10
(c) 38
(d) -7
(e) 64

Worked example

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = \frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{3}{4}$ and $\alpha\beta\gamma = \frac{2}{5}$ find:

(a) $(\alpha + 3)(\beta + 3)(\gamma + 3)$

(b) $(2 - \alpha)(2 - \beta)(2 - \gamma)$

Your turn

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = -\frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{4}$ and $\alpha\beta\gamma = -\frac{2}{5}$ find:

(a) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(b) $(1 - \alpha)(1 - \beta)(1 - \gamma)$

(a) $\frac{71}{10}$

(b) $\frac{53}{20}$

Worked example

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = \frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{3}{4}$ and $\alpha\beta\gamma = \frac{2}{5}$ find $(\alpha\beta)^3 + (\alpha\gamma)^3 + (\beta\gamma)^3$

Your turn

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = -\frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{4}$ and $\alpha\beta\gamma = -\frac{2}{5}$ find $(\alpha\beta)^3 + (\alpha\gamma)^3 + (\beta\gamma)^3$

$$\frac{723}{1600}$$

Worked example

The three roots of a cubic equation are α, β and γ.
Given that $\alpha\beta\gamma = 5$, $\alpha\beta + \beta\gamma + \gamma\alpha = -4$ and $\alpha + \beta + \gamma = 3$, find the value of
$(\alpha + 2)(\beta + 2)(\gamma + 2)$

Your turn

The three roots of a cubic equation are α, β and γ.
Given that $\alpha\beta\gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ and $\alpha + \beta + \gamma = 3$, find the value of
$(\alpha + 3)(\beta + 3)(\gamma + 3)$

43

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

(b) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(c) $\alpha^2\beta^2\gamma^2\delta^2$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

(b) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(c) $\alpha^3\beta^3\gamma^3\delta^3$

(a) $-\frac{3}{20}$

(b) $\frac{7}{4}$

(c) $\frac{64}{27}$

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find:

(a) $(\alpha + 2)(\beta + 2)(\gamma + 2)(\delta + 2)$

(b) $(2 - \alpha)(2 - \beta)(2 - \gamma)(2 - \delta)$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find:

(a) $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$

(b) $(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)$

(a) $\frac{113}{60}$

(b) $\frac{77}{60}$

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find

$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find

$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2$

$$\frac{823}{240}$$

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find

$$(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find

$$(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$$

$$\frac{51}{25}$$