4) Volumes of revolution

4.1) Volumes of revolution around the x-axis

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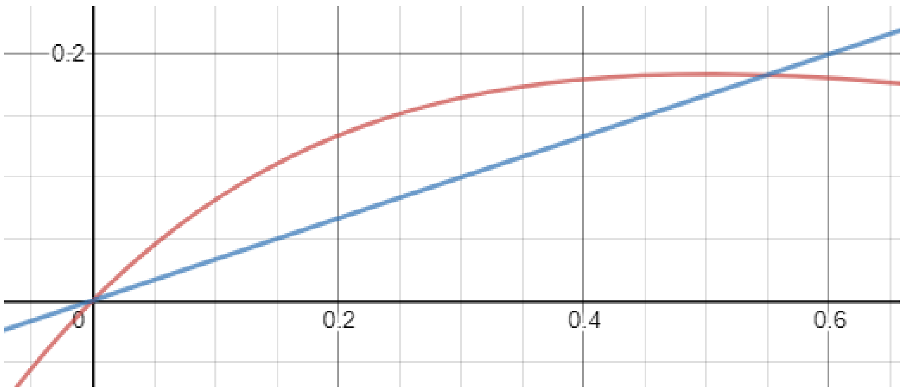
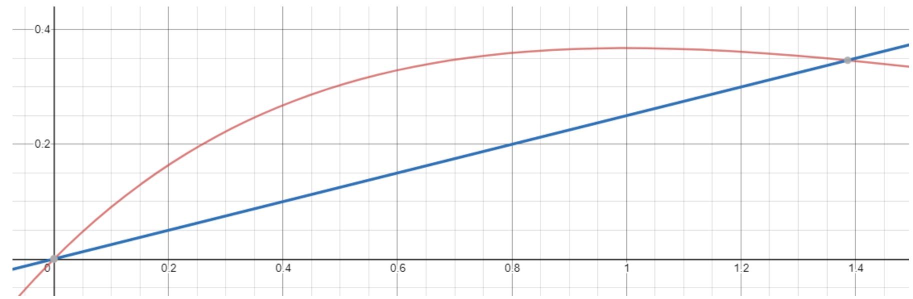
Worked example	Your turn
A finite region is bounded by the curve with equation $y = \sin 4x$, the x-axis and $x = \frac{\pi}{4}$. Find the volume of the solid formed when the region is rotated through 2π radians about the x-axis.	A finite region is bounded by the curve with equation $y = \sin 2x$, the x-axis and $x = \frac{\pi}{2}$. Find the volume of the solid formed when the region is rotated through 2π radians about the x-axis. $$\frac{\pi^2}{4}$$

Worked example	Your turn
A finite region is bounded by the curve with equation $y = 1 - 2\sin x$ $(0 < x < \pi)$ and the x-axis. Find the exact volume of the solid formed when the region is rotated through 2π radians about the x-axis.	A finite region is bounded by the curve with equation $y = 1 - 2\cos x$ $(0 < x < \pi)$ and the x-axis. Find the volume of the solid formed when the region is rotated through 2π radians about the x-axis. $\pi\left(4\pi + 3\sqrt{3}\right)$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the x-axis between the given limits: $$y = \sqrt{\frac{3 \sin x}{2+\cos x}} \text{ between } x = 0 \text{ and } x = \frac{\pi}{2}$$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the x-axis between the given limits: $$y = \sqrt{\frac{4 \sin x}{1+\cos x}} \text{ between } x = 0 \text{ and } x = \frac{\pi}{2}$$ $4\pi \ln 2$

Worked example	Your turn
Using integration by parts, find the exact volume of the solid generated when each curve is rotated through 2π radians about the x-axis between the given limits: $y = \frac{\sqrt{\ln x}}{x^2}$ between $x = 1$ and $x = 2$	Using integration by parts, find the exact volume of the solid generated when each curve is rotated through 2π radians about the x-axis between the given limits: $y = \sqrt{x}\sec x$ between $x = 0$ and $x = \frac{\pi}{4}$ $\frac{\pi}{4}(\pi - \ln 4)$

Worked example	Your turn
A finite region is bounded by the curve with equation $y = \frac{3}{10(2+5x)}$, the x-axis, and the lines $x = 1$ and $x = -2$. Find the exact volume of the solid formed when the region is rotated through $360°$ about the x-axis.	A finite region is bounded by the curve with equation $y = \frac{10}{3(5+2x)}$, the x-axis, and the lines $x = -1$ and $x = 2$. Find the exact volume of the solid formed when the region is rotated through $360°$ about the x-axis. $$\frac{100\pi}{81}$$

Worked example	Your turn
A finite region is bounded by the curve with equation $y = xe^{-2x}$ and the line $y = \frac{1}{3}x$ Find the volume of the solid formed when the region is rotated through 360° about the x-axis.	A finite region is bounded by the curve with equation $y = xe^{-x}$ and the line $y = \frac{1}{4}x$ Find the volume of the solid formed when the region is rotated through 360° about the x-axis.
	 0.237 (3 sf)

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4.2) Volumes of revolution around the y-axis

Worked example	Your turn
A finite region is bounded by the curve with equation $y = 8\ln x - 1$, the x-axis, the y-axis, and the line $y = 2$ Find the exact volume of the solid formed when the region is rotated by 2π radians about the y-axis.	A finite region is bounded by the curve with equation $y = 4\ln x - 1$, the x-axis, the y-axis, and the line $y = 4$ Find the exact volume of the solid formed when the region is rotated by 2π radians about the y-axis. $2\pi\sqrt{e}(e^2 - 1)$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $x = e^{y} - e^{-2y}$ between $y = 0$ and $y = 1$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $x = e^{2y} - e^{-y}$ between $y = 0$ and $y = 1$ $\frac{\pi}{4}(e^2 - 1)$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $x = \frac{\sqrt{4-\ln y}}{y}$ between $y = 1$ and $y = 4$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $x = \frac{\sqrt{5-\ln y}}{y}$ between $y = 1$ and $y = 5$ $\frac{\pi}{5}(\ln 5 + 16)$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $$y = \frac{2}{x} - 2 \text{ between } y = 0 \text{ and } y = 1$$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $$y = \frac{1}{x} - 1 \text{ between } y = 0 \text{ and } y = 1$$ $$\frac{\pi}{2}$$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $y = \frac{2-5x^2}{1-x^2}$ between $y = -1$ and $y = 1$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $y = \frac{5-2x^2}{x^2-1}$ between $y = -1$ and $y = 1$ $\pi(2 + 3\ln 3)$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $y = 3e^{x^2}$ between $y = 3$ and $y = 6$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $y = 2e^{x^2}$ between $y = 2$ and $y = 4$ $\pi(4\ln 2 - 2)$

Worked example	Your turn
Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $y = \text{arc}sin\sqrt{x}$ between $y = 0$ and $y = \frac{\pi}{2}$	Find the exact volume of the solid generated when each curve is rotated through 2π radians about the y-axis between the given limits: $y = \arccos\sqrt{x}$ between $y = 0$ and $y = \frac{\pi}{2}$ $\frac{3\pi^2}{16}$

Worked example	Your turn
A finite region is bounded b the curve with equation $x = \frac{1}{3y+1}$, the y-axis and the lines $y = 1$ and $y = b$. The region is rotated through 2π radians about the y-axis to generate a solid of revolution. Given that the volume of this solid is $\frac{\pi}{60}$, find b	A finite region is bounded b the curve with equation $x = \frac{1}{2y+1}$, the y-axis and the lines $y = 1$ and $y = b$. The region is rotated through 2π radians about the y-axis to generate a solid of revolution. Given that the volume of this solid is $\frac{\pi}{10}$, find b $$b = \frac{13}{4}$$

4.3) Volumes of revolution of parametrically defined curves

Worked example

The curve C has parametric equations

$x = t(1 - t), y = \frac{1}{1-t}, t \leq 0.$

The region R is bounded by C, the x-axis and the lines $x = 0$ and $x = -2$.

Find the exact volume of the solid formed when R is rotated 2π radians about the x-axis.

Your turn

The curve C has parametric equations

$x = t(1 + t), y = \frac{1}{1+t}, t \geq 0.$

The region R is bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.

Find the exact volume of the solid formed when R is rotated 2π radians about the x-axis.

$$\pi(2\ln 2 - \frac{1}{2})$$

Worked example	Your turn
A curve C has parametric equations $$x = \tan\theta\,, y = \sec^3\theta\,, 0 \leq \theta < \frac{\pi}{2}$$ A finite region is bounded by the curve C, the y-axis, and the lines $y = 1$ and $y = 8$. Find the exact volume of the solid formed when this region is rotated 2π radians about the y-axis.	A curve C has parametric equations $$x = \tan\theta\,, y = \sin\theta\,, 0 \leq \theta < \frac{\pi}{2}$$ A finite region is bounded by the curve C, the line $x = \sqrt{3}$ and the x-axis. Find the exact volume of the solid formed when this region is rotated 2π radians about the x-axis. $$\pi\sqrt{3} - \frac{1}{3}\pi^2$$

Worked example

A curve C has parametric equations

$$x = \frac{1}{3t}, y = \ln 3t\,, t \geq \frac{1}{3}$$

A finite region is bounded by the curve C, the x-axis, the y-axis and the line $y = a$.

Given that the volume of the solid formed when this region is rotated 2π radians about the y-axis is $\frac{12\pi}{25}$, find the exact value of a

Your turn

A curve C has parametric equations

$$x = \frac{1}{2t}, y = \ln 2t\,, t \geq \frac{1}{2}$$

A finite region is bounded by the curve C, the x-axis, the y-axis and the line $y = a$.

Given that the volume of the solid formed when this region is rotated 2π radians about the y-axis is $\frac{24\pi}{49}$, find the exact value of a

$$a = \ln 7$$

Worked example	Your turn
A curve C has parametric equations $$x = 2\cos t, y = t^2, \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$$ A finite region is bounded by the curve C and the y-axis. Find the exact volume of the solid formed when this region is rotated 2π radians about the y-axis	A curve C has parametric equations $$x = 2\sin t, y = t^2, 0 \leq t \leq \pi$$ A finite region is bounded by the curve C and the y-axis. Find the exact volume of the solid formed when this region is rotated 2π radians about the y-axis $2\pi^3$

4.4) Modelling with volumes of revolution

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Worked example	Your turn
A vase is modelled using a diagram. The maximum diameter of the vase on the diagram is $4\ cm$. The cross-section of the model is described by the curve with parametric equations $x = 2\sin 2t\,, y = 4\cos t + 2,\ 0 \le t \le \frac{\pi}{2}$, where the units of x and y are in cm. The vase is formed by rotating this curve about the y-axis to form a solid of revolution. (a) Find the volume of water required to fill the vase to a height of 3cm. The real goldfish bowl has a maximum diameter of 24cm. (b) Find the volume of water required to fill the real goldfish bowl to the corresponding height.	A goldfish bowl is modelled using a diagram. The diameter of the bowl on the diagram is $4\ cm$. The cross-section of the model is described by the curve with parametric equations $x = 2\sin t\,, y = 2\cos t + 2, \frac{\pi}{6} \le t \le \frac{11\pi}{6}$, where the units of x and y are in cm. The goldfish bowl is formed by rotating this curve about the y-axis to form a solid of revolution. (a) Find the volume of water required to fill the model to a height of 3cm. The real goldfish bowl has a maximum diameter of 48cm. (b) Find the volume of water required to fill the real goldfish bowl to the corresponding height. (a) $9\pi\ cm^3$ (b) $48900\ cm^3$ (3 sf)