4) Roots of polynomials

4.1) Roots of a quadratic equation

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Worked example

The roots of the quadratic equation $2x^2 + 5x + 4 = 0$ are α and β. Without solving the equation, find the values of:

(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\frac{1}{\alpha} + \frac{1}{\beta}$ (d) $\alpha^2 + \beta^2$

Your turn

The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β. Without solving the equation, find the values of:

(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\frac{1}{\alpha} + \frac{1}{\beta}$ (d) $\alpha^2 + \beta^2$

(a) $\frac{5}{2}$

(b) -2

(c) $-\frac{5}{4}$

(d) $\frac{41}{4}$

Worked example

The roots of the quadratic equation $6x^2 + 9x - 2 = 0$ are α and β. Without solving the equation, find the value of $\alpha^3 + \beta^3$

Your turn

The roots of the quadratic equation $6x^2 - 9x + 2 = 0$ are α and β. Without solving the equation, find the value of $\alpha^3 + \beta^3$

$$\frac{15}{8}$$

Worked example

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{3}{2}$ and $\beta = -\frac{5}{4}$.

Find integer values for a, b and c

Your turn

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$.

Find integer values for a, b and c

$$a = 8, b = 2, c = -15$$

Worked example

The equation $ax^2 + 6x + c = 0$, where a and c are real constants, has roots α and α^*. Given that $Re(\alpha) = 1$ and $Im(\alpha) = 4i$, find the values of a and c

Your turn

The equation $ax^2 + 8x + c = 0$, where a and c are real constants, has roots α and α^*. Given that $Re(\alpha) = 2$ and $Im(\alpha) = 3i$, find the values of a and c

$a = -2, c = -26$

4.2) Roots of a cubic equation

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Worked example

α, β and γ are the roots of the cubic equation $2x^3 - 3x^2 + 4x - 2 = 0$.

Without solving the equation, find the values of:

(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \gamma\alpha$ (c) $\alpha\beta\gamma$ (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Your turn

α, β and γ are the roots of the cubic equation $2x^3 + 3x^2 - 4x + 2 = 0$.

Without solving the equation, find the values of:

(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \gamma\alpha$ (c) $\alpha\beta\gamma$ (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(a) $-\frac{3}{2}$

(b) -2

(c) -1

(d) 2

Worked example

α, β and γ are the roots of the cubic equation $2x^3 - 3x^2 + 4x - 6 = 0$.
Without solving the equation, find the values of $\alpha^2\beta^2\gamma^2$ and $\alpha^3\beta^3\gamma^3$

Your turn

α, β and γ are the roots of the cubic equation $2x^3 + 3x^2 - 4x + 4 = 0$.
Without solving the equation, find the values of $\alpha^2\beta^2\gamma^2$ and $\alpha^3\beta^3\gamma^3$

$\alpha^2\beta^2\gamma^2 = 4$ and $\alpha^3\beta^3\gamma^3 = -8$

Worked example

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 3i, \beta = 1 + 3i$ and $\gamma = 3$.
Find integers values for a, b, c and d.

Your turn

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i, \beta = 1 + 2i$ and $\gamma = 2$.
Find integers values for a, b, c and d.

$$a = 1, b = -4, c = 9, d = -10$$

Worked example

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = \frac{2}{3}, \beta = \frac{1}{3}$ and $\gamma = 1$.

Find integers values for a, b, c and d.

Your turn

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$ and $\gamma = 1$.

Find integers values for a, b, c and d.

$$a = 4, b = -12, c = 11, d = -3$$

Worked example

The cubic equation $x^3 - 42x^2 + 336x - 512 = 0$ has roots $\alpha, k\alpha$, and $k^2\alpha$ for some real constant k. Find the values of α and k

Your turn

The cubic equation $x^3 - 14x^2 + 56x - 64 = 0$ has roots $\alpha, k\alpha$, and $k^2\alpha$ for some real constant k. Find the values of α and k

$\alpha = 2, k = 2$ or $\alpha = 8, k = \frac{1}{2}$

4.3) Roots of a quartic equation

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Worked example

α, β, γ, and δ are the roots of the quartic equation $x^4 - 3x^3 - 2x^2 + x - 4 = 0$

Find the values of:

(a) $\alpha + \beta + \gamma + \delta$

(b) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

(c) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

(d) $\alpha\beta\gamma\delta$

(e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

(f) $\alpha^2\beta^2\gamma^2\delta^2$

Your turn

α, β, γ, and δ are the roots of the quartic equation $x^4 + 3x^3 + 2x^2 - x + 4 = 0$

Find the values of:

(a) $\alpha + \beta + \gamma + \delta$

(b) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

(c) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

(d) $\alpha\beta\gamma\delta$

(e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

(f) $\alpha^3\beta^3\gamma^3\delta^3$

(a) -3

(b) 2

(c) 1

(d) 4

(e) $\frac{1}{4}$

(f) 64

Worked example

The roots of a quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

are $\alpha = \frac{2}{3}, \beta = \frac{1}{2}, \gamma = 2$ and $\delta = -\frac{3}{2}$

Find integer values of a, b and c

Your turn

The roots of a quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

are $\alpha = -\frac{3}{2}, \beta = -\frac{1}{2}, \gamma = -2$ and $\delta = \frac{2}{3}$

Find integer values of a, b and c

$$a = 12, b = 40, c = 25, d = -20, e = -12$$

Worked example

The equation $x^4 + 4x^3 + px^2 + qx - 80 = 0, x \in \mathbb{C}, p, q \in \mathbb{R}$, has roots $\alpha, \beta, \gamma, \delta$.
Given that $\gamma = -4 + 2i$ and $\delta = \gamma^*$.

(a) Show that $\alpha + \beta - 4 = 0$ and that $\alpha\beta + 4 = 0$

(b) Hence find all the roots of the quartic equation and find the values of p and q.

Your turn

The equation $x^4 + 2x^3 + px^2 + qx - 60 = 0, x \in \mathbb{C}, p, q \in \mathbb{R}$, has roots $\alpha, \beta, \gamma, \delta$.
Given that $\gamma = -2 + 4i$ and $\delta = \gamma^*$.

(a) Show that $\alpha + \beta - 2 = 0$ and that $\alpha\beta + 3 = 0$

(b) Hence find all the roots of the quartic equation and find the values of p and q.

(a) Shown

(b) $p = 9, q = -52$

4.4) Expressions relating to the roots of a polynomial

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Worked example

A quadratic equation has roots α and β. Given that $\alpha + \beta = 3$ and $\alpha\beta = 4$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ (b) $\alpha^2\beta^2$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$

Your turn

A quadratic equation has roots α and β. Given that $\alpha + \beta = 4$ and $\alpha\beta = 3$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ (b) $\alpha^2\beta^2$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$

(a) $\frac{4}{3}$
(b) 9
(c) 10
(d) 28

Worked example

A quadratic equation has roots α and β. Given that $\alpha + \beta = 3$ and $\alpha\beta = 4$, find:

(a) $(\alpha + 3)(\beta + 3)$

(b) $(\alpha^2 - 5)(\beta^2 - 5)$

Your turn

A quadratic equation has roots α and β. Given that $\alpha + \beta = 4$ and $\alpha\beta = 3$, find:

(a) $(\alpha + 5)(\beta + 5)$

(b) $(\alpha^2 - 3)(\beta^2 - 3)$

(a) 48

(b) -12

Worked example

A cubic equation has roots α, β and γ.
Given that $\alpha + \beta + \gamma = -2$, $\alpha\beta + \alpha\gamma + \beta\gamma = 3$ and $\alpha\beta\gamma = -4$ find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
(b) $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$
(d) $(\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2$
(e) $\alpha^2\beta^2\gamma^2$

Your turn

A cubic equation has roots α, β and γ.
Given that $\alpha + \beta + \gamma = 2$, $\alpha\beta + \alpha\gamma + \beta\gamma = -3$ and $\alpha\beta\gamma = 4$ find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
(b) $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$
(d) $(\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2$
(e) $\alpha^3\beta^3\gamma^3$

(a) $-\frac{3}{4}$
(b) 10
(c) 38
(d) -7
(e) 64

Worked example

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = \frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{3}{4}$ and $\alpha\beta\gamma = \frac{2}{5}$ find:

(a) $(\alpha + 3)(\beta + 3)(\gamma + 3)$

(b) $(2 - \alpha)(2 - \beta)(2 - \gamma)$

Your turn

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = -\frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{4}$ and $\alpha\beta\gamma = -\frac{2}{5}$ find:

(a) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(b) $(1 - \alpha)(1 - \beta)(1 - \gamma)$

(a) $\frac{71}{10}$

(b) $\frac{53}{20}$

Worked example

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = \frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{3}{4}$ and $\alpha\beta\gamma = \frac{2}{5}$ find $(\alpha\beta)^3 + (\alpha\gamma)^3 + (\beta\gamma)^3$

Your turn

A cubic equation has roots α, β and γ.

Given that $\alpha + \beta + \gamma = -\frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{4}$ and $\alpha\beta\gamma = -\frac{2}{5}$ find $(\alpha\beta)^3 + (\alpha\gamma)^3 + (\beta\gamma)^3$

$$\frac{723}{1600}$$

Worked example

The three roots of a cubic equation are α, β and γ.
Given that $\alpha\beta\gamma = 5$, $\alpha\beta + \beta\gamma + \gamma\alpha = -4$ and $\alpha + \beta + \gamma = 3$, find the value of
$(\alpha + 2)(\beta + 2)(\gamma + 2)$

Your turn

The three roots of a cubic equation are α, β and γ.
Given that $\alpha\beta\gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ and $\alpha + \beta + \gamma = 3$, find the value of
$(\alpha + 3)(\beta + 3)(\gamma + 3)$

43

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

(b) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(c) $\alpha^2\beta^2\gamma^2\delta^2$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find:

(a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

(b) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(c) $\alpha^3\beta^3\gamma^3\delta^3$

(a) $-\frac{3}{20}$

(b) $\frac{7}{4}$

(c) $\frac{64}{27}$

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find:

(a) $(\alpha + 2)(\beta + 2)(\gamma + 2)(\delta + 2)$

(b) $(2 - \alpha)(2 - \beta)(2 - \gamma)(2 - \delta)$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find:

(a) $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$

(b) $(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)$

(a) $\frac{113}{60}$

(b) $\frac{77}{60}$

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find

$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find

$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2$

$$\frac{823}{240}$$

Worked example

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = -\frac{1}{2}, \sum \alpha\beta = \frac{3}{4}, \sum \alpha\beta\gamma = \frac{1}{5}$ and $\alpha\beta\gamma\delta = -\frac{4}{3}$, find

$$(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$$

Your turn

A quartic equation has roots α, β, γ and δ

Given that $\sum \alpha = \frac{1}{2}, \sum \alpha\beta = -\frac{3}{4}, \sum \alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find

$$(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$$

$$\frac{51}{25}$$

4.5) Linear transformations of roots

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Worked example

The quadratic equation $x^2 + 3x - 10 = 0$ has roots α and β.
Without finding the roots, determine the equation with roots $\alpha - 2$ and $\beta - 2$

Your turn

The quadratic equation $x^2 - 3x - 10 = 0$ has roots α and β.
Without finding the roots, determine the equation with roots $\alpha - 1$ and $\beta - 1$

$w^2 - w - 12 = 0$

Worked example

The cubic equation $x^3 + 2x^2 - 3x + 4 = 0$ has roots α, β and γ. Find the equation of the polynomial with roots:

$3\alpha, 3\beta$ and 3γ

Your turn

The cubic equation $x^3 - 2x^2 + 3x - 4 = 0$ has roots α, β and γ. Find the equation of the polynomial with roots:

$2\alpha, 2\beta$ and 2γ

$w^3 - 4w^2 + 12w - 32 = 0$

Worked example

The cubic equation $x^3 + 2x^2 - 3x + 4 = 0$ has roots α, β and γ. Find the equation of the polynomial with roots:

$$\alpha - 2, \beta - 2 \text{ and } \gamma - 2$$

Your turn

The cubic equation $x^3 - 2x^2 + 3x - 4 = 0$ has roots α, β and γ. Find the equation of the polynomial with roots:

$$\alpha + 3, \beta + 3 \text{ and } \gamma + 3$$

$$w^3 - 11w^2 + 42w - 58 = 0$$

Worked example

The cubic equation $x^3 + 2x^2 - 4 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$

Your turn

The cubic equation $x^3 - 2x^2 + 4 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equation with roots $(3\alpha - 1)$, $(3\beta - 1)$, $(3\gamma - 1)$ and $(3\delta - 1)$

$$w^3 - 3w^2 - 9w + 103 = 0$$

Worked example

The quartic equation $x^4 + 3x^3 - x^2 - 15x - 1 = 0$ has roots α, β, γ and δ. Find the equation of the polynomial with roots:

$(3\alpha - 1)$, $(3\beta - 1)$, $(3\gamma - 1)$ and $(3\delta - 1)$

Your turn

The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots α, β, γ and δ. Find the equation of the polynomial with roots:

$(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$

$w^4 - 10w^3 + 24w^2 + 98w - 97 = 0$