Solving using partial fractions

We have already seen in Pure Year 2 how we can use partial fractions to integrate. We can use this to further expand our repertoire of integration techniques for expressions of the form $\frac{1}{a^2 \pm x^2}$ and $\frac{1}{\sqrt{a^2 \pm x^2}}$

Example

Prove that $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right| + c$

When you write as partial fractions, ensure you have the **most general possible non-top heavy fraction**, i.e. the ‘order’ (i.e. maximum power) of the numerator is **one less** than the denominator.

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Example

Show that $\int \frac{1+x}{x^3+9x}\,dx = A\ln\left(\frac{x^2}{x^2+9}\right) + B\arctan\left(\frac{x}{3}\right) + c$, where A and B are constants to be found.

If the fraction is top-heavy, you'll have a quotient. As per Pure Year 2, if the order of numerator and denominator is the same, you'll need an extra constant term. If the power is 1 greater in the numerator, you'll need a quotient of $Ax + B$, and so on.

$$\frac{4x^2+x}{x^2+x} = \frac{4x^2+x}{x(x+1)} = A + \frac{B}{x} + \frac{C}{x+1}$$

Test Your Understanding

(a) Express $\frac{x^4+x}{x^4+5x^2+6}$ as partial fractions.

(b) Hence find $\int \frac{x^4+x}{x^4+5x^2+6}\ dx.$

Ex 3D Pg 67-68