CP2, Chapter 3

Methods in Calculus

Course Structure

1. Improper Integrals
2. Mean Value of a Function
3. Differentiating and Integrating Inverse Trig Functions
4. Integrating using Partial Fractions

5 **Further calculus**	5.1	**Derive formulae for and calculate volumes of revolution.**	**Both** $\pi \int y^2 \, dx$ **and** $\pi \int x^2 \, dy$ **are required.** Students should be able to find a volume of revolution given either Cartesian equations or parametric equations.
	5.2	Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.	For example, $\int_0^{\infty} e^{-x} \, dx$, $\int_0^2 \frac{1}{\sqrt{x}} \, dx$
	5.3	Understand and evaluate the mean value of a function.	Students should be familiar with the mean value of a function f(x) as, $\frac{1}{b-a}\int_a^b f(x) \, dx$
	5.4	Integrate using partial fractions.	Extend to quadratic factors $ax^2 + c$ in the denominator

5 **Further calculus** *continued*	5.5	Differentiate inverse trigonometric functions.	For example, students should be able to differentiate expressions such as, $\arcsin x + x\sqrt{(1-x^2)}$ and $\frac{1}{2}\arctan x^2$
	5.6	Integrate functions of the form $(a^2 - x^2)^{-\frac{1}{2}}$ and $(a^2 - x^2)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.	

Improper Integrals

STARTER 1: Determine $\int_{-1}^{1} \frac{1}{x^2}\, dx$. Is there an issue?

STARTER 2: Determine $\int_{-1}^{1} \frac{1}{x^2}\, dx$. Is there an issue?

STARTER 3: Determine $\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx$. Is there an issue?

If a function $f(x)$ exists and is continuous for all values of x in the interval $[a, b]$ then the definite integral $\int_a^b f(x)\, dx$ represents the area enclosed by the curve $y = f(x)$, the x axis and the lines $x = a$ and $x = b$.

Here, we consider integrals where one or both of the limits are infinite, or where the function is not defined at some point within in the given interval. These are called improper integrals. In these cases, it is still possible for the function to enclose a finite area.

The integral $\int_a^b f(x)\,dx$ is improper if either:

- One or both of the limits is infinite
- $f(x)$ is undefined at $x = a,\ x = b$ are another point in the interval $[a, b]$.

If an improper integral exists it is said to be **convergent.** If it does not exist it is said to be **divergent.**

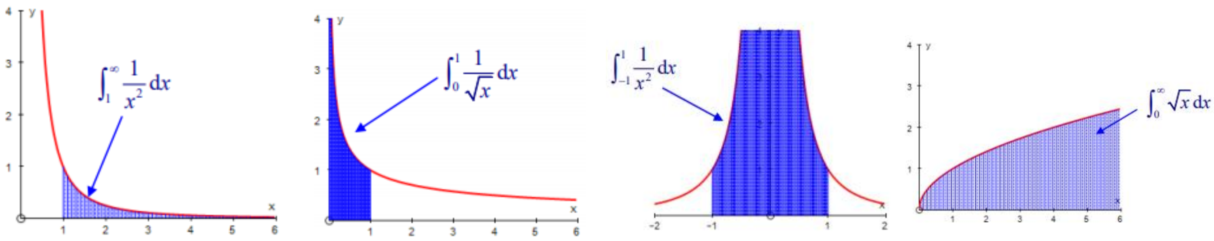
For example:

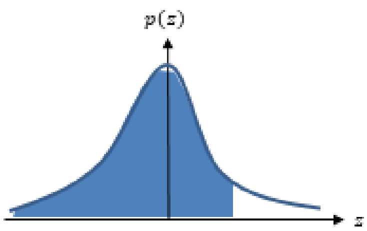
$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ is an improper integral since one of its limits is infinity;

$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is an improper integral since it is undefined at $x = 0$.

$\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x$ is an improper integral since it is undefined at $x = 0$;

$\int_0^\infty \sqrt{x}\,\mathrm{d}x$ is an improper integral since one of its limits is infinity;





We've seen use of improper integrals with the **normal distribution** (Stats Year 2), which has the probability function:

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}}$$

The z table determines the probability up to a particular value of z. So $\phi(z) = P(Z \le z) = \int_{-\infty}^{z} p(x)\,dx$. As the area under the whole graph is 1, the improper integral $\int_{-\infty}^{\infty} p(x)\,dx = 1$

We can't use ∞ in calculations directly. We can make use of the lim function we saw in differentiation by first principles.

To find $\int_a^\infty f(x)\,dx$, determine $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx$

Examples

1.Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ or show that it is not convergent.

2. Evaluate $\int_1^\infty \frac{1}{x}\,dx$ or show that it is not convergent.

<u>Undefined Values of $f(x)$</u>

We need to **avoid values** with the range $[a, b]$ **for which the expression is not defined**. But just as we avoided ∞ by considering the limit as $t \to \infty$, we can similarly find what the area converges to as x tends towards the undefined value.

Examples

1. Evaluate $\int_0^1 \frac{1}{x^2}\, dx$ or show that it is not convergent.

2. Evaluate $\int_0^2 \frac{x}{\sqrt{4-x^2}}\, dx$ or show that it is not convergent.

Further Examples

Find, if possible, the values of

(i) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(ii) $\int_{-1}^1 \frac{1}{x^2}\,dx$

Integrating between $-\infty$ and ∞

If both limits in the integral are infinite, then you need to split the integral into the sum of 2 improper integrals.

If both these integrals converge, then the original integral converges, but if either diverges, then the original integral is also divergent.

Example

(a) Find $\int xe^{-x^2}dx$

(b) Hence show that $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges and find its value.

Ex 3a, pages 56-58