3) Methods in calculus

3.1) Improper integrals

Chapter CONTENTS

Worked example	Your turn
Find the value of the improper integral $$\int_1^{\infty} \frac{1}{x^3}\,dx$$	Find the value of the improper integral $$\int_1^{\infty} \frac{1}{x^2}\,dx$$ 1

Worked example	Your turn
Find the value of the improper integral $$\int_{2}^{\infty} x^{-\frac{5}{2}}\,dx$$	Find the value of the improper integral $$\int_{2}^{\infty} x^{-\frac{3}{2}}\,dx$$ $$\sqrt{2}$$

Worked example	Your turn
Find the value of the improper integral $$\int_0^{\infty} e^{-2x}\,dx$$	Find the value of the improper integral $$\int_0^{\infty} e^{-3x}\,dx$$ $$\frac{1}{3}$$

Worked example	Your turn
Show that the integral does not converge: $$\int_0^1 \frac{1}{x^3}\,dx$$	Show that the integral does not converge: $$\int_0^1 \frac{1}{x^2}\,dx$$ Shown

Worked example	Your turn
Show that the integral does not converge: $$\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$$	Show that the integral does not converge: $$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$ Shown

Worked example	Your turn
Show that the integral converges and find its value: $$\int_{-\infty}^{\infty} x^2 e^{-x^3}\, dx$$	Show that the integral converges and find its value: $$\int_{-\infty}^{\infty} x e^{-x^2}\, dx$$ 0

Worked example	Your turn
Evaluate the integral: $$\int_0^2 \frac{6x}{\sqrt[3]{4-x^2}}\,dx$$	Evaluate the integral: $$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$$ 2

Worked example	Your turn
Show that the integral is divergent: $$\int_0^{\frac{\pi}{2}} \tan x \, dx$$	Show that the integral is divergent: $$\int_0^{\pi} \sec^2 x \, dx$$ Shown

Worked example	Your turn
Find the exact value of $$\int_0^\infty \frac{1}{3x^2+4x+1}\,dx$$	Find the exact value of $$\int_0^\infty \frac{1}{2x^2+3x+1}\,dx$$ $\ln 2$

3.2) The mean value of a function

Chapter CONTENTS

Worked example	Your turn
Find the mean value of $f(x) = \frac{3}{\sqrt{2+5x}}$ over the interval $[4,7]$	Find the mean value of $f(x) = \frac{4}{\sqrt{2+3x}}$ over the interval $[2,6]$ $\frac{4}{3}(\sqrt{5} - \sqrt{2})$

Worked example	Your turn
Find the mean value of $f(x) = \frac{2\sin x \cos x}{\cos 2x+3}$ over the interval $[0, \frac{\pi}{4}]$	Find the mean value of $f(x) = \frac{\sin x \cos x}{\cos 2x+2}$ over the interval $[0, \frac{\pi}{2}]$ $\frac{\ln 3}{2\pi}$

Worked example	Your turn
Find the mean value of $f(x) = \frac{5x}{2x^2-3x-2}$ over the interval $[1, 4]$	Find the mean value of $f(x) = \frac{5x}{2x^2+3x-2}$ over the interval $[1, 5]$ $\frac{1}{4}\ln\frac{49}{3}$

Worked example	Your turn
Find the mean value of $f(x) = \frac{1}{\sqrt{4-x}}$ over the interval $[0, 4]$	Find the mean value of $f(x) = \frac{1}{\sqrt{2-x}}$ over the interval $[0, 2]$ $\sqrt{2}$

Worked example	Your turn
$$f(x)=\frac{3}{1-e^x}$$ (a) Find the mean value of $f(x)$ over the interval $[\ln 2, \ln 10]$ (b) Use your answer to part a to find the mean value over the interval $[\ln 2, \ln 10]$ of $f(x)+5$. (c) Use geometric considerations to write down the mean value of $-f(x)$ over the interval $[\ln 2, \ln 10]$	$$f(x)=\frac{4}{1+e^x}$$ (a) Find the mean value of $f(x)$ over the interval $[\ln 2, \ln 6]$ (b) Use your answer to part a to find the mean value over the interval $[\ln 2, \ln 6]$ of $f(x)+4$. (c) Use geometric considerations to write down the mean value of $-f(x)$ over the interval $[\ln 2, \ln 6]$ (a) $\frac{4\ln\frac{9}{7}}{\ln 3}$ (b) $\frac{4\ln\frac{9}{7}}{\ln 3}+4$ (c) $-\frac{4\ln\frac{9}{7}}{\ln 3}$

3.3) Differentiating inverse trigonometric functions

Chapter CONTENTS

Worked example	Your turn
Find:	Find:
$$\frac{d}{dx}(\arcsin x)$$	$$\frac{d}{dx}(\arccos x)$$
	$$-\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x)$$	

Worked example	Your turn
Find: $\frac{d}{dx}(\arccos x^4)$ $\frac{d}{dx}(\arctan 2x^3)$	Find: $\frac{d}{dx}(\arcsin x^2)$ $\frac{2x}{\sqrt{1-x^4}}$

Worked example	Your turn
Find: $$\frac{d}{dx}(\operatorname{arcsec} 3x)$$	Find: $$\frac{d}{dx}(\operatorname{arcsec} 2x)$$ $$\frac{1}{x\sqrt{4x^2-1}}$$

Worked example	Your turn
Given that $y = \arctan\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$	Given that $y = \arctan\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$ $-\frac{1}{1+x^2}$

Worked example	Your turn
Prove: $$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$	Prove: $$\sin(\operatorname{arcsec} x) = \sqrt{1 - \frac{1}{x^2}}$$ Proof

3.4) Integrating with inverse trigonometric functions

Chapter CONTENTS

Worked example	Your turn
Use an appropriate substitution to show that: $$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$ $$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$	Use an appropriate substitution to show that: $$\int \frac{1}{1-x^2}\,dx = \arcsin x + C$$ Proof

Worked example	Your turn
Use an appropriate substitution to show that: $$\int \frac{1}{a^2 - x^2} dx = \arcsin\frac{x}{a} + C$$ where a is a positive constant and $\|x\| < a$	Use an appropriate substitution to show that: $$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C, a > 0, \|x\| < a$$ Proof

Worked example	Your turn
Find: $$\int \frac{3}{2+x^2}\,dx$$	Find: $$\int \frac{4}{5+x^2}\,dx$$ $$\frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)+c$$

Worked example	Your turn
Find: $$\int \frac{1}{49+16x^2}\,dx$$	Find: $$\int \frac{1}{25+9x^2}\,dx$$ $$\frac{1}{15}\arctan\left(\frac{3x}{5}\right)+c$$

Worked example	Your turn
Find: $$\int_{-1}^{2} \frac{1}{\sqrt{21-3x^2}}\,dx$$	Find: $$\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx$$ $$\frac{\pi}{6}$$

Worked example	Your turn
Find: $$\int \frac{x+9}{\sqrt{1-9x^2}}\,dx$$	Find: $$\int \frac{x+4}{\sqrt{1-4x^2}}\,dx$$ $$\frac{1}{4}\sqrt{1-4x^2} + 2\arcsin 2x + c$$

Worked example	Your turn
Find: $$\int \frac{6x-5}{3+x^2}\,dx$$	Find: $$\int \frac{8x-3}{4+x^2}\,dx$$ $$4\ln(4+x^2)-\frac{3}{2}\arctan\left(\frac{x}{2}\right)+c$$

Worked example	Your turn
Find: $$\int \frac{3x-1}{\sqrt{5-4x^2}}\,dx$$	Find: $$\int \frac{4x-1}{\sqrt{6-5x^2}}\,dx$$ $$-\frac{4}{5}\sqrt{6-5x^2}-\frac{1}{\sqrt{5}}\arcsin\left(\sqrt{\frac{5}{6}}x\right)+c$$

3.5) Integrating using partial fractions

Worked example	Your turn
Find: $$\int \frac{1+x}{x^3+16x}\,dx$$	Find: $$\int \frac{1+x}{x^3+9x}\,dx$$ $$\frac{1}{18}\ln\left(\frac{x^2}{x^2+9}\right)+\frac{1}{3}\arctan\left(\frac{x}{3}\right)+c$$

Worked example	Your turn
Find: $$\int \frac{3x - x^2}{(x^2+9)(x+3)}\, dx$$	Find: $$\int \frac{x^2 - 3x}{(x^2+6)(x+2)}\, dx$$ $$\ln\lvert x+2\rvert - \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{6}}\right) + c$$

Worked example	Your turn
Find: $$\int \frac{x^4 + x}{x^4 + 7x^2 + 12}\, dx$$	Find: $$\int \frac{x^4 + x}{x^4 + 5x^2 + 6}\, dx$$ $$x + \frac{1}{2}\ln\left|\frac{x^2 + 2}{x^2 + 3}\right| + 2\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right) - 3\sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right) + c$$

Worked example	Your turn
Evaluate: $$\int_0^1 \frac{2}{(x+2)(x^2+2)}$$	Find: $$\int_0^1 \frac{2}{(x+1)(x^2+1)}$$ $$\frac{1}{4}(\pi + 2\ln 2)$$

Worked example	Your turn
Find: $$\int \frac{x^4+1}{x(x^2+3)^2}\,dx$$	Find: $$\int \frac{x^4+1}{x(x^2+2)^2}\,dx$$ $$\frac{1}{4}\ln\lvert x\rvert+\frac{3}{8}\ln\lvert x^2+2\rvert+\frac{5}{4(x^2+2)}+c$$

Worked example	Your turn
Find: $$\int \frac{1}{3x^2+6x+7}\,dx$$	Find: $$\int \frac{1}{2x^2+4x+11}\,dx$$ $$\frac{1}{3\sqrt{2}}\arctan\left(\frac{\sqrt{2}(x+1)}{3}\right)+c$$