Given that $f(x) = x^2 e^{-x}$

Show that $f'''(x) = (6x - 6 - x^2)e^{-x}$

Why we might represent functions as power series?

Example:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Recall that a **polynomial** is an expression of the form $a + bx + cx^2 + \cdots$, i.e. a sum of x terms with non-negative integer powers. A **power series** is an infinitely long polynomial.

1 **Not all functions can be integrated**. For example, there is no way in which we can write $\int x^x\,dx$ in terms of standard mathematical functions. $\int e^{x^2}\,dx$ is another well known example of a function we can't integrate. Since the latter is used in the probability (density) function of a Normal Distribution, and we'd have to integrate to find the cumulative distribution function, this explains why it's not possible to calculate z-values on a calculator. However, **polynomials integrate very easily**. We can approximate integrals to any arbitrary degree of accuracy by replacing the function with its power series.

2 **It allows us/calculators to find irrational numbers in decimal form to an arbitrary amount of accuracy**.
Since for example $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$ then plugging in 1, we get $\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ and hence $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$*. We can similarly find e.

3 It allows us to find approximate solutions to more difficult differential equations.

* This is a poor method of generating digits of π though: because the denominator is only increasing by 2 each time, it converges very slowly! There are better power series where the denominator has a much higher growth rate and hence the terms become smaller quicker.

Maclaurin Expansion

Let's assume that any function can be written as an infinite sum of terms in the form ax^n

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots + a_rx^r + \ldots$$

Our aim is simply to find the values of a_r
This can be done using differentiation and substituting $x = 0$
Why does this work?

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

Find the Maclaurin series for e^x.

https://www.desmos.com/calculator/xwixhh31da

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

Find the Maclaurin series for $\sin x$.

https://www.desmos.com/calculator/5nclbbrmni

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

Find the Maclaurin series for $\cos x$.

https://www.desmos.com/calculator/rxbov4dk36

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for all } x$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \frac{x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \quad (-1 \leqslant x \leqslant 1)$$

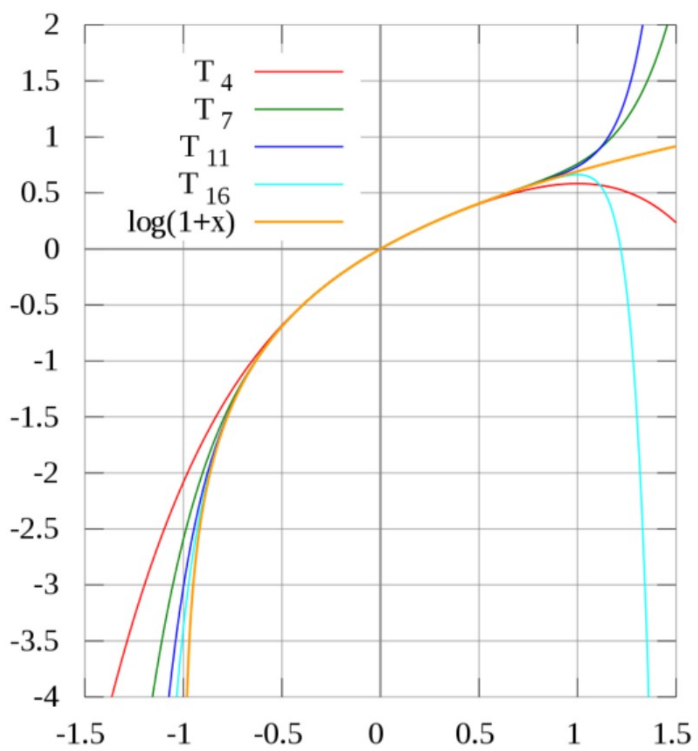
$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

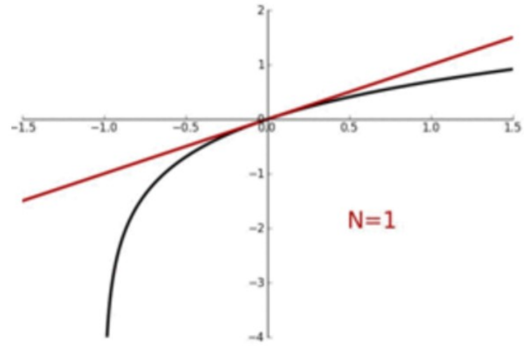
(a) Find the Maclaurin series for $\ln(1+x)$.

(b) Using only the first three terms of the series in (a), find estimates for (i) $\ln 1.05$ (ii) $\ln 1.25$ (iii) $\ln 1.8$

What do you notice about the estimates?
Why is this happening?

https://www.desmos.com/calculator/geaurexlsq





We previously said that we can only guarantee the curve is the same, i.e. the expansion is valid, 'around' $x = 0$.
For e^x and $\cos x$ we got lucky in that the curve turned out to be the same everywhere, for all x.

But as per the animation above, we can see that away from $x = 0$, the curve actually gets worse with more terms in the expansion!
Looking at the graph on the left, for what range of x is this expansion valid for?

$$-\mathbf{1} \leq \boldsymbol{x} < \mathbf{1}$$

Ex 2C

Composite Functions

✏ Standard Expansions (given in formula booklet)

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^r}{r!} + \cdots$ (valid for all x)

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{r-1}\frac{x^r}{r} + \cdots$ $-1 < x \leq 1$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \cdots$ (valid for all x)

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^r x^{2r}}{2r!} + \cdots$ (valid for all x)

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^r x^{2r+1}}{2r+1} - \cdots$ $-1 \leq x \leq 1$

We can also apply these when the input to the function is different.

$\cos(2x^2) =$