Chapter 2b - Series (Year 2)
Maclaurin Expansion

1:: Method of differences

Given that

$$(2r+1)^3 = Ar^3 + Br^2 + Cr + 1,$$

(*a*) find the values of the constants A, B and C. **(2)**

(*b*) Show that

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2.$$ **(2)**

(*c*) Using the result in part (*b*) and the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$ **(5)**

2:: Maclaurin Expansions

Representing functions as a power series (i.e. infinitely long polynomial)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

3:: Maclaurin Expansions for Composite Functions

"Given that terms in $x^n, n > 4$ may be neglected, use the series for e^x and $\sin x$ to show that $e^{\sin x} \approx 1 + x + \frac{x^2}{2}$"

Higher Derivatives

	Original Function	**1st Derivative**	**2nd Derivative**	**nth Derivative**
Lagrange's Notation	$f(x)$	$f'(x)$	$f''(x)$	$f^{(n)}(x)$
Leibniz's Notation	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^ny}{dx^n}$
Newton's Notation	y	$\dot{y}$	$\ddot{y}$	$\overset{n}{\dot{y}}$

Given that $y = \ln(1-x)$, find the value of $\frac{d^3y}{dx^3}$ when $x = \frac{1}{2}$.

$f(x) = e^{x^2}$.

(a) Show that $f'(x) = 2x\, f(x)$

(b) By differentiating the result in part a twice more with respect to x, show that:

(i) $f''(x) = 2f(x) + 2x\, f'(x)$

(ii) $f'''(x) = 2xf''(x) + 4f'(x)$

(c) Deduce the values of $f'(0), f''(0), f'''(0)$

We will see the point of finding $f'(0), f''(0), f'''(0)$ in the next section.

Given that $f(x) = x^2e^{-x}$

Show that $f'''(x) = (6x - 6 - x^2)e^{-x}$

Why we might represent functions as power series?

Example: $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Recall that a **polynomial** is an expression of the form $a + bx + cx^2 + \cdots$, i.e. a sum of x terms with non-negative integer powers. A **power series** is an infinitely long polynomial.

1 **Not all functions can be integrated**. For example, there is no way in which we can write $\int x^x\,dx$ in terms of standard mathematical functions. $\int e^{x^2}\,dx$ is another well known example of a function we can't integrate. Since the latter is used in the probability (density) function of a Normal Distribution, and we'd have to integrate to find the cumulative distribution function, this explains why it's not possible to calculate z-values on a calculator. However, **polynomials integrate very easily**. We can approximate integrals to any arbitrary degree of accuracy by replacing the function with its power series.

2 **It allows us/calculators to find irrational numbers in decimal form to an arbitrary amount of accuracy**.
Since for example $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$ then plugging in 1, we get $\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ and hence $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$*. We can similarly find e.

3 It allows us to find approximate solutions to more difficult differential equations.

* This is a poor method of generating digits of π though: because the denominator is only increasing by 2 each time, it converges very slowly! There are better power series where the denominator has a much higher growth rate and hence the terms become smaller quicker.