Chapter 2a - Series (Year 2)
Method of Differences

1:: Method of differences

Given that

$$(2r+1)^3 = Ar^3 + Br^2 + Cr + 1,$$

(*a*) find the values of the constants A, B and C. **(2)**

(*b*) Show that

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2.$$ **(2)**

(*c*) Using the result in part (*b*) and the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$ **(5)**

2:: Maclaurin Expansions

Representing functions as a power series (i.e. infinitely long polynomial)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

3:: Maclaurin Expansions for Composite Functions

"Given that terms in x^n, $n > 4$ may be neglected, use the series for e^x and $\sin x$ to show that $e^{\sin x} \approx 1 + x + \frac{x^2}{2}$"

Method of Differences

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \cdots \times \frac{n-1}{n}$$

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

$$\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

Hint: Perhaps write out the first few terms?

✏ If $u_n = f(n) - f(n+1)$ then

$$\sum_{r=1}^{n} u_r = f(1) - f(n+1)$$

Known as 'method of differences'.

Show that

$$4r^3 = r^2(r+1)^2 - (r-1)^2r^2$$

Hence prove, by the method of differences that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Exam Note: Exam questions usually have two parts:

(a) Showing some expression is equivalent to one in form $f(n) - f(n+1)$

(b) Using method of differences to simplify summation.

Partial Fraction Example

First split into partial fractions in hope that we get form $f(r) - f(r+1)$ or something similar.

Find $\sum_{r=1}^{n} \frac{1}{4r^2-1}$ using the method of differences.

(a) Express $\frac{2}{(2r+1)(2r+3)}$ in partial fractions.

(b) Using your answer to (a), find, in terms of n,

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)}$$

Give your answer as a single fraction in its simplest form.

Ex 2A

Leave out 3, 8, 9, 12, 13

Harder Ones: $f(r) - f(r+2)$

If $u_n = f(n) - f(n+2)$ then...

$$\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$

(a) Express $\frac{2}{(r+1)(r+3)}$ in partial fractions.

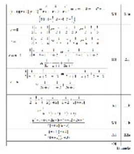
(b) Hence show that:

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

(c) Evaluate $\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)}$, giving your answer to 5 decimal places.

Ex 2A
Q3, 8, 9, 12, 13

An example in an atypical form



4. Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

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1. (*a*) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

(1)

(*b*) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}$.

(5)

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6. (*a*) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

(2)

(*b*) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{n(an+b)}{4(n+1)(n+2)},$$

where a and b are constants to be found.

(6)

(*c*) Hence show that

$$\sum_{r=n+1}^{2n}\frac{1}{r(r+2)}=\frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}.$$

(3)

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

2. A company operating a coal mine is concerned about the mine running out of coal. It is estimated that 2.5 million tonnes of coal are left in the mine. The company wishes to mine all of this coal in 20 years.

In order to mine the coal in a regulated manner, the company models the amount of coal to be mined in the coming years by the formula

$$M_r = \frac{10}{r^2 + 8r + 15}$$

where M_r is the amount of coal, in millions of tonnes, mined in year r, with the first year being year 1

(a) Show that, according to the model, the total amount of coal, in millions of tonnes, mined in the first n years is given by

$$T_n = \frac{9n^2 + 41n}{k(n+4)(n+5)}$$

where k is a constant to be determined.

(6)

(b) Explain why, according to this model, the mine will never run out of coal.

(2)

The company decides to mine an extra fixed amount each year so that all the coal will be mined in exactly 20 years.

(c) Refine the formula for M_r so that 2.5 million tonnes of coal will be exhausted in exactly 20 years of mining.

(2)