2) Series

2.1) The method of differences

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Worked example

Show that $r = \frac{1}{2}(r(r+1) - r(r-1))$

Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Your turn

Show that

$$4r^3 = r^2(r+1)^2 - (r-1)^2 r^2$$

Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Shown

Worked example	Your turn
Find, using the method of differences, $$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$$	Find, using the method of differences, $$\sum_{r=1}^{n} \frac{1}{r(r+1)}$$ $$\frac{n}{n+1}$$

Worked example	Your turn
Find, using the method of differences, $$\sum_{r=1}^{n} \frac{2}{r(r+2)}$$	Find, using the method of differences, $$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)}$$ $$\frac{n(5n+13)}{6(n+2)(n+3)}$$

Worked example

Find, using the method of differences,

$$\sum_{r=1}^{n} \frac{2}{(4r^2 + 8r + 3)}$$

Your turn

Find, using the method of differences,

$$\sum_{r=1}^{n} \frac{2}{4r^2 - 1}$$

$$\frac{2n}{2n+1}$$

Worked example	Your turn
Find the value of $$\sum_{r=100}^{200} \frac{4}{(4r-1)(4r+3)}$$ to 4 decimal places	Find the value of $$\sum_{r=16}^{25} \frac{4}{(2r+1)(2r+5)}$$ to 4 decimal places 0.0218 (4 dp)

2.2) Higher derivatives

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Worked example	Your turn
Given that $y = \ln(1+x)$, find the value of $\frac{d^3y}{dx^3}$ when $x = \frac{1}{2}$	Given that $y = \ln(1-x)$, find the value of $\frac{d^3y}{dx^3}$ when $x = \frac{1}{2}$ -16

Worked example	Your turn
Given that $y = \sec 3x$, find the value of $\frac{d^3y}{dx^3}$ when $x = \frac{\pi}{4}$	Given that $y = \sin^2 3x$, find the value of $\frac{d^4y}{dx^4}$ when $x = \frac{\pi}{6}$ 648

Worked example	Your turn
$f(x) = \ln\left(x + \sqrt{1 + x^2}\right)$ (a) Show that $(1 + x^2)f'''(x) + 3xf''(x) + f'(x) = 0$ (b) Deduce the values of $f'(0), f''(0), f'''(0)$	$f(x) = e^{x^2}$ (a) Show that: (i) $f'(x) = 2x\, f(x)$ (ii) $f''(x) = 2f(x) + 2x\, f'(x)$ (iii) $f'''(x) = 2xf''(x) + 4f'(x)$ (b) Deduce the values of $f'(0), f''(0), f'''(0)$ (a) Shown (b) $f(0) = 1$ $f'(0) = 0$ $f''(0) = 2$ $f'''(0) = 0$

2.3) Maclaurin series

Worked example	Your turn
Find the Maclaurin series for $\frac{1}{1-x}$	Find the Maclaurin series for $\sqrt{1+x}$ $$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

Worked example	Your turn
Find the Maclaurin series for $\ln(1+x)$	Find the Maclaurin series for e^x $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$

Worked example	Your turn
Find the Maclaurin series for $\cos^2 x$ up to and including the term in x^4	Find the Maclaurin series for $\sin^2 x$ up to and including the term in x^4 $x^2 - \frac{x^4}{3} + \cdots$

Worked example	Your turn
(a) Find the Maclaurin series for $\cos x$ (b) Use the first three terms of the series to find an approximation for $\cos 30°$	(a) Find the Maclaurin series for $\sin x$ (b) Use the first two terms of the series to find an approximation for $\sin 10°$ (a) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots$ (b) 0.17365 (5 dp)

2.4) Series expansions of compound functions

Worked example	Your turn
Find the Maclaurin series for e^{4x} up to and including the term in x^4	Find the Maclaurin series for e^{3x} up to and including the term in x^4 $e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \cdots$

Worked example	Your turn
Find the Maclaurin series for $\ln(1+3x)$ up to and including the term in x^4	Find the Maclaurin series for $\ln(1+2x)$up to and including the term in x^4 $\ln(1+2x) = 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \cdots$

Worked example	Your turn
Write down the first four non-zero terms in the series expansion, in ascending powers of x, of $$\sin(4x^3)$$	Write down the first four non-zero terms in the series expansion, in ascending powers of x, of $$\cos(2x^2)$$ $$1 - 3x^4 + \frac{2}{3}x^8 - \frac{4}{45}x^{12} + \cdots$$

Worked example	Your turn
Find the first three non-zero terms of the series expansion of $\ln\left(\frac{\sqrt{1+3x}}{1-2x}\right)$, and state the interval in x for which the expansion is valid.	Find the first three non-zero terms of the series expansion of $\ln\left(\frac{\sqrt{1+2x}}{1-3x}\right)$, and state the interval in x for which the expansion is valid. $4x+\frac{7}{2}x^2+\frac{31}{3}x^3+\cdots$ Valid for $-\frac{1}{3}\leq x<\frac{1}{3}$

Worked example	Your turn
Find the first three terms in the Maclaurin series expansion of $e^{\cos x}$	Find the first three terms in the Maclaurin series expansion of $e^{\sin x}$ $$1 + x + \frac{x^2}{2}$$

Worked example	Your turn
Find the series expansions, up to and including the term in x^4, of: $$\ln(1+2x-3x^2)$$ State the range of values of x for which the expansion is valid	Find the series expansions, up to and including the term in x^4, of: $$\ln(1+x-2x^2)$$ State the range of values of x for which the expansion is valid $$x-\frac{5x^2}{2}+\frac{7x^3}{3}-\frac{17x^4}{4}+\cdots$$ $$-\frac{1}{2}<x\leq\frac{1}{2}$$

Worked example	Your turn
Find the series expansions, up to and including the term in x^4, of: $$\ln(16+8x+x^2)$$ State the range of values of x for which the expansion is valid	Find the series expansions, up to and including the term in x^4, of: $$\ln(9+6x+x^2)$$ State the range of values of x for which the expansion is valid $$2\ln 3+\frac{2x}{3}-\frac{x^2}{9}+\frac{2x^3}{81}-\frac{x^4}{162}+\cdots$$ $$-3<x\leq 3$$

Worked example	Your turn
Find the series expansion, up to and including the term in x^4, of: $$\ln(16 + 8x + x^2)$$ State the range of values of x for which the expansion is valid	Find the series expansion, up to and including the term in x^4, of: $$\ln(9 + 6x + x^2)$$ State the range of values of x for which the expansion is valid $$2\ln 3 + \frac{2x}{3} - \frac{x^2}{9} + \frac{2x^3}{81} - \frac{x^4}{162} + \cdots$$ $$-3 < x \leq 3$$

Worked example	Your turn
Using the first two terms, $x - \frac{x^3}{3}$, in the expansion of $\arctan x$, find the first four terms of $e^{\arctan x}$ Deduce the first four terms in the series expansion of $e^{-\arctan x}$	Using the first two terms, $x + \frac{x^3}{3}$, in the expansion of $\tan x$, find the first four terms of $e^{\tan x}$ Deduce the first four terms in the series expansion of $e^{-\tan x}$ $$e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \cdots$$ $$e^{-\tan x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \cdots$$