2) Argand diagrams

2.1) Argand diagrams

Chapter CONTENTS

Worked example	Your turn
Plot on an Argand diagram:	Plot on an Argand diagram:
$4+3i$	$8+6i$
$-1+i$	$1-i$
$-2i$	-2
$-1-3i$	$-1+3i$

Worked example	Your turn
Plot on an Argand diagram: $4 + 3i$ $3i - 4$ $3 - 4i$	Plot on an Argand diagram: $-4i + 3$

2.2) Modulus and argument

Chapter CONTENTS

Worked example	Your turn
Determine the modulus and argument: $4 + 3i$	Determine the modulus and argument: $8 + 6i$ Modulus $= 10$ Argument $= 0.644$ (3 sf)
$-1 + i$	$1 - i$ Modulus $= \sqrt{2}$ Argument $= -\frac{\pi}{4}$
$-2i$	-2 Modulus $= 2$ Argument $= \pm\pi$
$-1 - 3i$	$-1 + 3i$ Modulus $= \sqrt{10}$ Argument $= 1.25$ (3 sf)

Worked example	Your turn
$z = 3 - 2i$ Find: a) z^2 b) $\lvert z^2 \rvert$ c) $\arg(z^2)$ d) Show z and z^2 on an Argand diagram	$z = 2 - 3i$ Find: a) z^2 b) $\lvert z^2 \rvert$ c) $\arg(z^2)$ d) Show z and z^2 on an Argand diagram (a) $-5 - 12i$ (b) 13 (c) -1.97 (3 sf) (d) Shown

Worked example	Your turn
$w = 2 + 3i$ Given that $\arg(\lambda + 5i + w) = \frac{\pi}{4}$, where λ is a real constant, find the value of λ	$w = 2 + 5i$ Given that $\arg(\lambda + 3i + w) = \frac{\pi}{4}$, where λ is a real constant, find the value of λ $\lambda = 6$

Worked example	Your turn
The complex numbers w and z are given by $w = k - i$ and $z = 3 - 5ki$, where k is a real constant. Given that $\arg(w + z) = \frac{\pi}{3}$, find the exact value of k	The complex numbers w and z are given by $w = k + i$ and $z = -4 + 5ki$, where k is a real constant. Given that $\arg(w + z) = \frac{2\pi}{3}$, find the exact value of k $k = \dfrac{21\sqrt{3} - 17}{22}$

Worked example	Your turn
The complex numbers w and z are defined such that $\arg w = \frac{\pi}{20}$, $\|w\| = 3$ and $\arg z = \frac{7\pi}{20}$. Given that $\arg(w+z) = \frac{\pi}{4}$, find the value of $\|z\|$	The complex numbers w and z are defined such that $\arg w = \frac{\pi}{10}$, $\|w\| = 5$ and $\arg z = \frac{2\pi}{5}$. Given that $\arg(w+z) = \frac{\pi}{5}$, find the value of $\|z\|$ 2.63 (3 sf)

2.3) Modulus-argument form of complex numbers

Worked example	Your turn
Express $z = -1 + i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$	Express $z = -1 - i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$ $r = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

Worked example	Your turn
Express $z = -\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$	Express $z = -1 - \sqrt{3}\,i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$ $r = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

Worked example	Your turn
The complex number z is such that $\|z\| = 3$ and $\arg z = \frac{\pi}{4}$. Find z in the form $a + bi$, where a and b are exact real numbers to be found.	The complex number z is such that $\|z\| = 5$ and $\arg z = \frac{3\pi}{4}$. Find z in the form $a + bi$, where a and b are exact real numbers to be found. $a = -\frac{5\sqrt{2}}{2}, b = \frac{5\sqrt{2}}{2}$
The complex number z is such that $\|z\| = 4$ and $\arg z = -\frac{3\pi}{4}$. Find z in the form $a + bi$, where a and b are exact real numbers to be found.	

Worked example	Your turn
$z_1 = 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$ $z_2 = 3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$	$z_1 = 8(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10})$ $z_2 = 4(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})$
Find:	Find:
i) $\lvert z_1 z_2 \rvert$	i) $\lvert z_1 z_2 \rvert$ 32
ii) $\arg(z_1 z_2)$	ii) $\arg(z_1 z_2)$ $-\frac{\pi}{2}$
iii) $z_1 z_2$ in the form $r(\cos\theta + i\sin\theta)$	iii) $z_1 z_2$ in the form $r(\cos\theta + i\sin\theta)$ $32\left(\cos\left(-\frac{\pi}{2}\right) + i\sin(-\frac{\pi}{2})\right)$
iv) $z_1 z_2$ in the form $x + iy$	iv) $z_1 z_2$ in the form $x + iy$ -32i

Worked example	Your turn
$z_1 = 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$ $z_2 = 3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$	$z_1 = 8(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10})$ $z_2 = 4(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})$
Find:	Find:
i) $\left\|\frac{z_1}{z_2}\right\|$	i) $\left\|\frac{z_1}{z_2}\right\|$ 2
ii) $\arg(\frac{z_1}{z_2})$	ii) $\arg(\frac{z_1}{z_2})$ $-\frac{\pi}{10}$
iii) $\frac{z_1}{z_2}$ in the form $r(\cos\theta + i\sin\theta)$	iii) $\frac{z_1}{z_2}$ in the form $r(\cos\theta + i\sin\theta)$ $2\left(\cos\left(-\frac{\pi}{10}\right) + i\sin(-\frac{\pi}{10})\right)$
iv) $\frac{z_1}{z_2}$ in the form $x + iy$	iv) $\frac{z_1}{z_2}$ in the form $x + iy$ $1.90 - 0.618\text{i}$

Worked example	Your turn
$z_1 = 6(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12})$ $z_2 = 3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$	$z_1 = 8(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10})$ $z_2 = 4(\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5})$
Find:	Find:
i) $\lvert z_1 z_2\rvert$	i) $\lvert z_1 z_2\rvert$ 32
ii) $\arg(z_1 z_2)$	ii) $\arg(z_1 z_2)$ $-\frac{\pi}{10}$
iii) $z_1 z_2$ in the form $r(\cos\theta + i\sin\theta)$	iii) $z_1 z_2$ in the form $r(\cos\theta + i\sin\theta)$ $32\left(\cos\left(-\frac{\pi}{10}\right) + i\sin(-\frac{\pi}{10})\right)$
iv) $z_1 z_2$ in the form $x + iy$	iv) $z_1 z_2$ in the form $x + iy$ $30.4 - 9.89\text{i}$

2.4) Loci in the Argand diagram

Chapter CONTENTS

Worked example

Sketch the locus of points represented by

$$|z| = 3$$

$$|z| = 4$$

Your turn

Sketch the locus of points represented by

$$|z| = 5$$

Circle centre $(0, 0)$ radius 5

Cartesian equation: $x^2 + y^2 = 25$

Worked example	Your turn
Sketch the locus of points represented by $$\|z+3-5i\| = 2$$ and find its Cartesian equation	Draw the locus of points that satisfy: $$\|z-5-3i\| = 6$$ and find its Cartesian equation Circle centre $(5, 3)$ radius 6 $(x-5)^2 + (y-3)^2 = 36$
Sketch the locus of points represented by $$\|z-3+5i\| = 4$$ and find its Cartesian equation	

Worked example	Your turn
Sketch the locus of points represented by $$\|3 - z\| = 5$$ and find its Cartesian equation	Draw the locus of points that satisfy: $$\|3 - 2i - z\| = 3$$ and find its Cartesian equation Circle centre $(3, -2)$ radius 3 $(x - 2)^2 + (y + 2)^2 = 9$
Sketch the locus of points represented by $$\|2i - z\| = 4$$ and find its Cartesian equation	
Sketch the locus of points represented by $$\|2 - 3i - z\| = 2$$ and find its Cartesian equation	

Worked example	Your turn
A complex number z is represented by the point P. Given that $\|z-3+5i\|=2$ (a) Sketch the locus of P (b) Find the Cartesian equation of the locus. (c) Find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$ (d) Find the minimum and maximum values of $\|z\|$	A complex number z is represented by the point P. Given that $\|z-5-3i\|=3$ (a) Sketch the locus of P (b) Find the Cartesian equation of the locus. (c) Find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$ (d) Find the minimum and maximum values of $\|z\|$ (a) Circle centre $(5, 3)$ radius 3 (b) $(x-5)^2+(y-3)^2=36$ (c) 1.08 (3 sf) (d) Max $\|z\|=\sqrt{34}+3$ Min $\|z\|=\sqrt{34}-3$

Worked example	Your turn
Sketch the locus of points represented by $$\|z\| = \|z + 4i\|$$ and find its Cartesian equation	Sketch the locus of points represented by $$\|z\| = \|z - 6i\|$$ and find its Cartesian equation Perpendicular bisector of $(0, 0)$ and $(0, 6)$ $y = 3$
Sketch the locus of points represented by $$\|z\| = \|z - 5\|$$ and find its Cartesian equation	

Worked example	Your turn
Sketch the locus of points represented by $$\lvert z-3i\rvert = \lvert z+1\rvert$$ and find its Cartesian equation	Sketch the locus of points represented by $$\lvert z-3\rvert = \lvert z+i\rvert$$ and find its Cartesian equation Perpendicular bisector of $(3,0)$ and $(0,-1)$ $y=-3x+4$

Worked example	Your turn
Find the Cartesian equation of the locus of z if $\|z-3i\| = \|z+1\|$ and sketch the locus of z on an Argand diagram. Hence, find the least possible value of $\|z\|$.	Find the Cartesian equation of the locus of z if $\|z-3\| = \|z+i\|$, and sketch the locus of z on an Argand diagram. Hence, find the least possible value of $\|z\|$. $\frac{2\sqrt{10}}{5}$

Worked example	Your turn
Given that the complex number z satisfies the equation $\|z - 8 + 6i\| = 5$, find the minimum value of $\|z\|$ and the maximum.	Given that the complex number z satisfies the equation $\|z - 12 - 5i\| = 3$, find the minimum value of $\|z\|$ and the maximum. Minimum $= 10$ Maximum $= 16$

Worked example	Your turn
Sketch the locus of points represented by $$\arg(z) = \frac{\pi}{4}$$ and find its Cartesian equation	Sketch the locus of points represented by $$\arg(z) = \frac{\pi}{6}$$ and find its Cartesian equation Half-line from origin $(0, 0)$ $y = \frac{1}{\sqrt{3}}x, \qquad x > 0, y > 0$
Sketch the locus of points represented by $$\arg(z) = \frac{\pi}{3}$$ and find its Cartesian equation	

Worked example	Your turn
Sketch the locus of points represented by $$\arg(z-2+3i)=\frac{\pi}{4}$$ and find its Cartesian equation	Sketch the locus of points represented by $$\arg(z+3+2i)=\frac{3\pi}{4}$$ and find its Cartesian equation Half-line from $(-3,-2)$ $y=-x-5, \quad x<-3, y>-2$

Worked example	Your turn
Find the complex number z which satisfies both $\|z+3-2i\| = 50$ and $\arg(z+3-2i) = \frac{\pi}{4}$	Find the complex number z which satisfies both $\|z+3+2i\| = 10$ and $\arg(z+3+2i) = \frac{3\pi}{4}$ $z = \left(-3-5\sqrt{2}\right) + i\left(-2+5\sqrt{2}\right)$

Worked example	Your turn
If the complex number z satisfies both $\arg z = \frac{\pi}{4}$ and $\arg(z-3) = \frac{\pi}{2}$, (a) Find the value of z (b) Hence, find $\arg(z-6)$	If the complex number z satisfies both $\arg z = \frac{\pi}{3}$ and $\arg(z-4) = \frac{\pi}{2}$, (a) Find the value of z (b) Hence, find $\arg(z-8)$ (a) $z = 4 + 4\sqrt{3}i$ (b) $\frac{2\pi}{3}$

Worked example	Your turn
Given $\|z+4-8i\|=3$, show that the maximum value of $\arg(z+12-5i)$ in the interval $(-\pi,\pi)$ is $2\arcsin\left(\frac{3}{\sqrt{73}}\right)$	Given $\|z+8-4i\|=2$, show that the maximum value of $\arg(z+15-2i)$ in the interval $(-\pi,\pi)$ is $2\arcsin\left(\frac{2}{\sqrt{53}}\right)$ Shown

2.5) Regions in the Argand diagram

Chapter CONTENTS

Worked example	Your turn
On an Argand diagram, shade the region for which $$\|z-3+5i\| \leq 4$$	On an Argand diagram, shade the region for which $$\|z+3-5i\| \leq 2$$ Inside of solid-lined circle, centre $(3, -5)$, radius 2

Worked example	Your turn
On an Argand diagram, shade the region for which $$2 \leq \|z - 3 + 5i\| \leq 4$$ $$2 < \|z - 3 - 5i\| \leq 4$$	On an Argand diagram, shade the region for which $$2 \leq \|z + 3 - 5i\| < 4$$ Region enclosed between two circles. One solid-lined circle centred $(-3, 5)$ radius 2 One dotted-lined circle centred $(-3, 5)$ radius 4

Worked example	Your turn
On an Argand diagram, shade the region for which $$\lvert z-3\rvert < \lvert z-5\rvert$$ $$\lvert z-3i\rvert > \lvert z+5\rvert$$	On an Argand diagram, shade the region for which $$\lvert z+3\rvert < \lvert z-5i\rvert$$ Dotted line perpendicular bisector of $(-3, 0)$ and $(0, 5)$. Shaded below the line

Worked example	Your turn
On an Argand diagram, shade the region for which $\{z \in \mathbb{C}: \lvert z-4\rvert \le \lvert z-8-6i\rvert\} \cap \left\{z \in \mathbb{C}: 0 \le \arg(z-2-4i) \le \frac{\pi}{4}\right\}$	On an Argand diagram, shade the region for which $\{z \in \mathbb{C}: \lvert z-2\rvert \le \lvert z-6-8i\rvert\} \cap \left\{z \in \mathbb{C}: 0 \le \arg(z-4-2i) \le \frac{\pi}{2}\right\}$ Shaded region in first quadrant enclosed by half lines $x = 4$ and $y = 2$ both extending from $(4, 2)$ and perpendicular bisector of $(2, 0)$ and $(6, 8)$ $y = -\frac{1}{2}x + 6$

Worked example

On an Argand diagram, shade the region for which

$$0 \leq \arg(z - 3 - 5i) \leq \frac{\pi}{4}$$

$$\arg(z - 3 + 5i) > \frac{\pi}{2}$$

Your turn

On an Argand diagram, shade the region for which

$$0 \leq \arg(z + 3 - 5i) \leq \frac{\pi}{3}$$

Shaded between two solid half-lines.

First half-line horizontal from point $(3, -5)$ in 4^{th} quadrant only

Second half-line from point $(3, -5)$ at angle of $\frac{\pi}{3}$ to the horizontal