Applications of De'Moivre's Theorem 2: Roots

De Moivre's theorem also holds true for rational powers. We can use De Moivre's to solve equations of the form $z^n = w$, where $z, w \in \mathbb{C}$. This is equivalent to finding the nth roots of w

The fundamental theorem of algebra holds true for complex numbers:

Hence $z^n = w$, where $z, w \in \mathbb{C}$ has n distinct roots.

If w = 1 we call these roots of unity.

To find the roots of a complex equation we use the fact that the argument of a complex number is not unique:

If $z^n = r^n(cosn\theta + isinn\theta)$ then $z = r(\cos(\theta + 2K\pi) + isin(\theta + 2k\pi))$

Roots of Unity Example

Solve $z^3 = 1$

Method 1: By factorising

Method 2: De Moivre's

Notice:

- The first root will always be 1 since $1^n = 1$

- We add $\frac{2k\pi}{n}$ to the argument but leave the modulus unchanged e.g. when $n = 3$ we rotate the line $\frac{2\pi}{3}$ each time. When $k = n$ we have rotated $\frac{2\pi n}{n} = 2\pi$ so we get back to where we started.

- The first root is $z_1 = 1$, call the second root $z_2 = \omega$. Then…

$$\omega = r\left(\cos\left(\frac{2\pi}{n}\right) + isin\left(\frac{2\pi}{n}\right)\right) \qquad r = 1$$

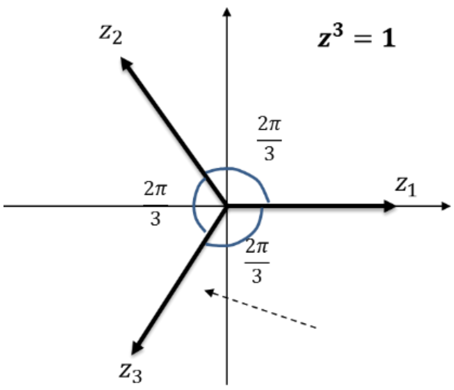
Consider $\omega^2 =$

$=$

$=$

What do you notice?

What would z_3 be?



The roots of $z^n = 1$ can be represented as $1,\ \omega,\ \omega^2, where\ \omega = e^{\frac{2\pi i}{n}}$.
Since the resultant 'vector' is 0, then $1 + \omega + \omega^2 = 0$

These roots form the vertices of a regular n-gon and all lie on a circle, radius r.

General nth Roots: $z^n = w, w \neq 1$

We can use a similar method when w is not equal to 1. Again, our first step is to write w in mod-arg form and consider multiples of the argument.

Example

Solve $\boldsymbol{z^4 = 2 + 2\sqrt{3}\, i}$

Test your understanding

a) Express the complex number $-2 + (2\sqrt{3})\mathrm{i}$ in the form $\mathrm{r}(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \leq \pi$.

(3)

b) Solve the equation

$$z^4 = -2 + (2\sqrt{3})\mathrm{i}$$

giving the roots in the form $\mathrm{r}(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \leq \pi$. (5)

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