Sum of Series

We can extend our knowledge of geometric series into complex numbers, where the same formulae hold true.

For $w, z \in \mathbb{C}$,

$$\Sigma_{r=0}^{n-1} wz^r = w + wz + wz^2 + \cdots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$$

$$\Sigma_{r=0}^{\infty} wz^r = w + wz + wz^2 + \cdots + wz^{n-1} = \frac{w}{z-1}$$

provided $|z| < 1$

Remember:

$$z^n - z^{-n} = 2i \sin n\theta$$

Thus if we had an expression of the form $e^{i\theta} - 1$, we could cleverly factorise out $e^{\frac{i\theta}{2}}$ (i.e. half the power) to get

$$e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}\right)$$

$$\rightarrow e^{\frac{i\theta}{2}}\left(2i \sin\left(\frac{\theta}{2}\right)\right)$$

Thus where $e^{i\theta} - 1$ occurs in a fraction, multiply numerator and denominator by $e^{-\frac{i\theta}{2}}$ so that we have just $e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}$

Example

Given that $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, where n is a positive integer, show that

$$1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\left(\frac{\pi}{2n}\right)$$

Practise the factorising………

$$\frac{3}{e^{2i\theta}-1} =$$

$$\frac{4e^{i\theta}}{e^{4i\theta}-1} =$$

$$\frac{1}{e^{\frac{i\theta}{3}}-1} =$$

Using mod-arg form to split summations

$\boldsymbol{e^{i\theta}+e^{2i\theta}+e^{3i\theta}+\cdots+e^{ni\theta}}$ **is a geometric series,**

$$\therefore \boldsymbol{e^{i\theta}+e^{2i\theta}+e^{3i\theta}+\cdots+e^{ni\theta} = \frac{e^{i\theta}\left(e^{ni\theta}-1\right)}{e^{i\theta}-1}}$$

Converting each exponential term to modulus-argument form would allow us to consider the real and imaginary parts of the series separately:

Example

$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{8i\theta}$, for $\theta \neq 2n\pi$, where n is an integer.

(a) Show that $S = \dfrac{e^{\frac{9i\theta}{2}} \sin 4\theta}{\sin\frac{\theta}{2}}$

Let $P = \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos 8\theta$ and $Q = \sin\theta + \sin 2\theta + \cdots + \sin 8\theta$

(b) Use your answer to part a to show that $P = \cos\frac{9\theta}{2} \sin 4\theta \ \ cosec\frac{\theta}{2}$ and find similar expressions for Q and $\frac{Q}{P}$