# Chapter 1

# Complex Numbers

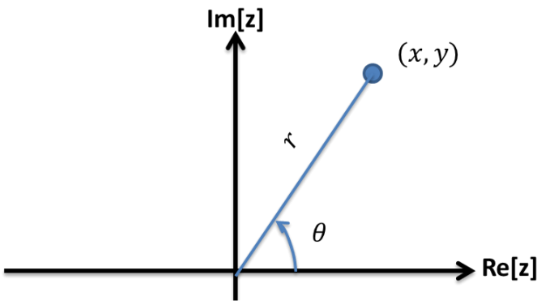
Chapter Overview

1. Exponential form of a complex number
2. Multiplying and dividing complex numbers
3. De Moivre's Theorem
4. De Moivre's for Trigonometric Identities
   a) Expressing $\cos n\theta$ / $\sin n\theta$ in terms of powers of $\cos\theta$
   b) Finding expressions for $\sin^n \theta$ and $\cos^n \theta$
5. Roots
6. Sums of series

| | | | |
|---|---|---|---|
| **2**<br>**Complex numbers**<br>*continued* | 2.8 | Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series. | To include using the results, $z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$ to find $\cos p\theta$, $\sin q\theta$ and $\tan r\theta$ in terms of powers of $\sin\theta$, $\cos\theta$ and $\tan\theta$ and powers of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of multiple angles.<br>For sums of series, students should be able to show that, for example,<br>$1 + z + z^2 + \ldots + z^{n-1} = 1 + \mathrm{i}\cot\left(\frac{\pi}{2n}\right)$<br>where $z = \cos\left(\frac{\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\pi}{n}\right)$ and $n$ is a positive integer. |
| | 2.9 | Know and use the definition $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and the form $z = r\mathrm{e}^{\mathrm{i}\theta}$ | Students should be familiar with<br>$\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})$ and<br>$\sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})$ |
| | 2.10 | Find the *n* distinct $n$th roots of $r\mathrm{e}^{\mathrm{i}\theta}$ for $r \neq 0$ and know that they form the vertices of a regular $n$-gon in the Argand diagram. | |
| | 2.11 | Use complex roots of unity to solve geometric problems. | |

Recap: Mod/ arg form

$$\boldsymbol{z = x + iy}$$




$r = |z| =$

$\theta = \arg(z) =$

$\boldsymbol{x} =$

$\boldsymbol{y} =$

$\boldsymbol{z = x + iy} \quad =$

| $\boldsymbol{x + iy}$ | $\boldsymbol{r}$ | $\boldsymbol{\theta}$ | **Mod-arg form** |
|---|---|---|---|
| $-1$ | | | |
| $i$ | | | |
| $1 + i$ | | | |
| $-\sqrt{3} + i$ | | | |

# Exponential Form

We've seen the Cartesian form a complex number $z = x + yi$ and the modulus-argument form $z = r(\cos\theta + i\sin\theta)$. But wait, there's a third form!

In the later chapter on Taylor expansions, you'll see that you that you can write functions as an infinitely long polynomial:

$$\cos x = 1 \quad - \frac{x^2}{2!} \quad + \frac{x^4}{4!} \quad - \frac{x^6}{6!} \quad +$$

$$\sin x \quad = x \quad - \frac{x^3}{3!} \quad + \frac{x^5}{5!} \quad - \cdots$$

$$e^x = \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} +$$

It looks like the $\cos x$ and $\sin x$ somehow add to give $e^x$. The one problem is that the signs don't quite match up.

**Exponential form** $\quad z = re^{i\theta}$

| $x + iy$ | **Mod-arg form** | **Exp Form** |
|---|---|---|
| $-1$ | | |
| $2 - 3i$ | | |
| | $\sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ | |
| | | $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ |
| | | $z = 2e^{\frac{23\pi i}{5}}$ |

<u>Example</u>

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

**Example 1**

Prove that $1 - e^{i\theta}\cos\theta = -ie^{i\theta}\sin\theta$.

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