1) Complex numbers

1.1) Imaginary and complex numbers

Chapter CONTENTS

Worked example	Your turn
Write in terms of i:	Write in terms of i:
$\sqrt{-39}$	$\sqrt{-49}$
	$7i$
$\sqrt{-40}$	$\sqrt{-20}$
	$(2\sqrt{5})i$

Worked example	Your turn
Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$: $(2 + 5i) + (3 + 4i)$ $(2 - 5i) - (4 - 3i)$	Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$: $(2 + 3i) + (4 + 5i)$ $6 + 8i$ $(2 - 3i) - (4 - 5i)$ $-2 + 2i$

Worked example	Your turn
Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$: $2(3 + 4i)$ $-5(6 - 7i)$	Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$: $-8(9 + 10i)$ $-72 - 80i$

Worked example	Your turn
Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$: $$\frac{6 - 8i}{2}$$ $$\frac{-7 + 21i}{7}$$	Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$: $$\frac{15 - 12i}{3}$$ $5 - 4i$

Worked example	Your turn
Given that $z_1 = a + 2i, z_2 = -3 + bi$, and $z_2 - z_1 = 5 + 7i$, find a and b, where $a, b \in \mathbb{R}$	Given that $z_1 = a + 5i, z_2 = -2 + 7i$, and $z_2 - z_1 = 3 + 11i$, find a and b, where $a, b \in \mathbb{R}$ $a = -5, b = 16$

Worked example	Your turn
Given that $z = a + bi$, and $w = a - bi$, where $a, b \in \mathbb{R}$, show that: $z + w$ is always real	Given that $z = a + bi$, and $w = a - bi$, where $a, b \in \mathbb{R}$, show that: $z - w$ is always imaginary $(a + bi) - (a - bi)$ $= a + bi - a + bi$ $= 2bi$ $= (2b)i$

Worked example	Your turn
Solve: $$z^2 = -9$$ $$z^2 + 16 = 0$$	Solve: $$z^2 + 25 = 0$$ $$z = \pm 5i$$

Worked example	Your turn
Solve: $(z+2)^2+9=0$	Solve: $(z+4)^2+25=0$
$(z-3)^2+16=0$	$z=-4\pm 5i$

Worked example	Your turn
Solve: $$z^2 + 4z + 13 = 0$$ $$z^2 - 6z + 25 = 0$$	Solve: $$z^2 + 8z + 41 = 0$$ $$z = -4 \pm 5i$$

Worked example	Your turn
Solve: $$z^2 + 3z + 13 = 0$$ $$3z^2 - 7z + 25 = 0$$	Solve: $$2z^2 - 8z + 41 = 0$$ $$z = 2 \pm \frac{\sqrt{66}}{2} i$$

Worked example	Your turn
The equation $z^2 + bz + 31 = 0$, where $b \in \mathbb{R}$, has distinct, non-real complex roots. Find the range of possible values of b	The equation $z^2 + bz + 13 = 0$, where $b \in \mathbb{R}$, has distinct, non-real complex roots. Find the range of possible values of b $-2\sqrt{13} < b < 2\sqrt{13}$

1.2) Multiplying complex numbers

Chapter CONTENTS

Worked example	Your turn
Determine the value of:	Determine the value of:
i^2	i^{10}
	-1
i^3	i^7
	$-i$
i^4	i^{40}
	1
$(3i)^5$	$(2i)^5$
	$32i$

Worked example	Your turn
Determine the value of: i^{101} i^{202} i^{3003}	Determine the value of: i^{10007} $-i$

Worked example	Your turn
Express each of the following in the form $a+bi$, where a, b are integers:	Express each of the following in the form $a+bi$, where a, b are integers:
$(2+3i)(2-3i)$	$(4+5i)(4-5i)$
	29
$(2+3i)(3+2i)$	$(4+5i)(5+4i)$
	$41i$
$(2-3i)^2$	$(4-5i)^2$
	$41-40i$

Worked example	Your turn
Simplify, giving your answer in the form $a + bi$: $$(1+i)^3$$ $$(1+i)^4$$	Simplify, giving your answer in the form $a + bi$: $$(1+i)^5$$ $$-4-4i$$

Worked example	Your turn
Given that $(a + 5i)(1 + bi) = 38 - 16i$, find the possible values of a and b	Given that $(a + 5i)(1 + bi) = 22 - 16i$, find the values of a and b $a = 7, b = -3$ $a = 15\ , b = -\frac{7}{5}$

1.3) Complex conjugation

Chapter CONTENTS

Worked example	Your turn
Write the complex conjugate for: $z = 2 + 3i$ $z = -2 - 3i$ $z = 3i - 2$	Write the complex conjugate for: $z = -5 - 4i$ $z^* = -5 + 4i$

Worked example	Your turn
Write in the form $a + bi$: $$\frac{5+4i}{2+3i}$$ $$\frac{2-3i}{4-5i}$$	Write in the form $a + bi$: $$\frac{5+4i}{2-3i}$$ $$-\frac{2}{13}+\frac{23}{13}i$$

Worked example	Your turn
Given that $z_1 = 2 + 3i$, $z_2 = \frac{5-12i}{z_1}$, find z_2 in the form $a + ib$, where a and b are real	Given that $z_1 = 3 + 2i$, $z_2 = \frac{12-5i}{z_1}$, find z_2 in the form $a + ib$, where a and b are real $2 - 3i$

Worked example	Your turn
Given that $z_1 = p - 3i$, $z_2 = 2 - 5i$, and that p is an integer, find $\frac{z_1}{z_2}$ in the form $a + ib$, where a and b are rational and given in terms of p	Given that $z_1 = p - 5i$, $z_2 = 2 + 3i$, and that p is an integer, find $\frac{z_1}{z_2}$ in the form $a + ib$, where a and b are rational and given in terms of p $$\frac{2p - 15}{13} + \frac{-10 - 3p}{13}i$$

Worked example

$$z = \frac{p+2i}{p-5i}, p \in \mathbb{R}, p > 0$$

Given that the real part of z is $\frac{6}{41}$, find the value of p

Your turn

$$z = \frac{p+3i}{p-7i}, p \in \mathbb{R}, p > 0$$

Given that the real part of z is $\frac{2}{37}$, find the value of p

$$p = 5$$

Worked example	Your turn
Given that $z = x + iy$, where $x, y \in \mathbb{R}$, find the value of x and y such that: $$(3 - i)z^* + 2iz = -9 - 13i$$ where z^* is the complex conjugate of z	Given that $z = x + iy$, where $x, y \in \mathbb{R}$, find the value of x and y such that: $$(3 - i)z^* + 2iz = 9 - i$$ where z^* is the complex conjugate of z $x = 5, y = 2$

1.4) Roots of quadratic equations

Worked example	Your turn
Given that $\alpha = 5 + 3i$ is one of the roots of a quadratic equation with real coefficients, (a) state the value of the other root, β. (b) find the quadratic equation.	Given that $\alpha = 7 + 2i$ is one of the roots of a quadratic equation with real coefficients, (a) state the value of the other root, β. (b) find the quadratic equation. (a) $\beta = 7 - 2i$ (b) $z^2 - 14z + 53 = 0$

Worked example	Your turn
Given that $\alpha = 5 + qi$ is one of the roots of the equation $z^2 - 5pz + 41 = 0$, where p and q are positive real constants, find the value of p and the value of q	Given that $\alpha = 5 + qi$ is one of the roots of the equation $z^2 - 2pz + 61 = 0$, where p and q are positive real constants, find the value of p and the value of q $p = 5\,, q = 6$

1.5) Solving cubic and quartic equations

Worked example	Your turn
Given that -2 is a root of the cubic equation $z^3 - 2z^2 - 3z + k = 0$ (a) Find the value of k (b) Find the other two roots	Given that -1 is a root of the cubic equation $z^3 - z^2 + 3z + k = 0$ (a) Find the value of k (b) Find the other two roots (a) $k = 5$ (b) $1 + 2i$ and $1 - 2i$

Worked example	Your turn
Given that $3+i$ is a root of the quartic equation $2z^4 - 37z^3 + 221z^2 - 380z - 250 = 0$, solve the equation completely.	Given that $3+i$ is a root of the quartic equation $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$, solve the equation completely. $z_1 = -5$ $z_2 = \frac{1}{2}$ $z_3 = 3 + i$ $z_4 = 3 - i$

Worked example	Your turn
Show that $z^2 + 9$ is a factor of $$z^4 - 8z^3 + 26z^2 - 72z + 153$$ Hence solve the equation $$z^4 - 8z^3 + 26z^2 - 72z + 153 = 0$$	Show that $z^2 + 4$ is a factor of $$z^4 - 2z^3 + 21z^2 - 8z + 68$$ Hence solve the equation $$z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$$ $z_1 = 2i$ $z_2 = -2i$ $z_3 = 1 + 4i$ $z_4 = 1 - 4i$

Worked example	Your turn
Given that 5 and $4+3i$ are roots of the equation $$x^3-13x^2+cx+d=0 \quad c,d\in\mathbb{R}$$ (a) Write down the other complex root (b) Find the value of c and the value of d	Given that 2 and $5+2i$ are roots of the equation $$x^3-12x^2+cx+d=0 \quad c,d\in\mathbb{R}$$ (a) Write down the other complex root (b) Find the value of c and the value of d (a) $5-2i$ (b) $c=49, d=-58$

Worked example	Your turn
Solve: $$z^4 = 1$$ $$z^4 = 16$$	Solve: $$z^4 = 81$$ $z_1 = 3$ $z_2 = -3$ $z_3 = 3i$ $z_4 = 1 - 4i$

Worked example	Your turn
$f(z) = z^3 + 4z^2 + kz + 36, k \in \mathbb{R}$ Given that $f(3i) = 0$, find the value of k and the other two roots of the equation	$f(z) = z^3 + 3z^2 + kz + 48, k \in \mathbb{R}$ Given that $f(4i) = 0$, find the value of k and the other two roots of the equation $k = 16$ $-4i$ and -3

Worked example	Your turn
Find the square root of $3 + 4i$	Find the square root of $5 + 12i$ $3 - 2i, -3 + 2i$
Find the square root of i	