CP1 Chapter 7

Linear Transformations

Chapter Overview

1. Use matrices to represent linear transformations

2. Use matrices to represent reflections, rotations (about the origin).

3. Invariant Lines and Points

4. Use matrices to represent enlargements.

5. Carry out successive transformations using matrix products.

6. Use inverse matrices to represent reverse transformations.

3.3	**Use matrices to represent linear transformations in 2-D.** **Successive transformations.** **Single transformations in 3-D.**	**For 2-D, identification and use of the matrix representation of single and combined transformations from: reflection in coordinate axes and lines $y = \pm x$, rotation through any angle about $(0, 0)$, stretches parallel to the x-axis and y-axis, and enlargement about centre $(0, 0)$, with scale factor k, $(k \neq 0)$, where $k \in \mathbb{R}$.** **Knowledge that the transformation represented by $\mathbf{AB}$ is the transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$.** **3-D transformations confined to reflection in one of $x = 0$, $y = 0$, $z = 0$ or rotation about one of the coordinate axes.** **Knowledge of 3-D vectors is assumed.**
3.4	**Find invariant points and lines for a linear transformation.**	**For a given transformation, students should be able to find the coordinates of invariant points and the equations of invariant lines.**

Linear Transformations

We can use matrices to describe linear transformations. A linear transformation moves all points $(\boldsymbol{x}, \boldsymbol{y})$ in a plane according to some rule.

Transforming a point $\begin{pmatrix} x \\ y \end{pmatrix}$ simply involves multiplying it by some matrix.

From above we can see that multiplying by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the mapping T: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$.

Example

1. Express the linear transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 4x + y \\ x - 2y \end{pmatrix}$ as a matrix.

2. Find matrices to represent these linear transformations.

a) $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y + x \\ 3x \end{pmatrix}$

b) $V: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -2y \\ 3x + y \end{pmatrix}$

3. A square has coordinates $(1,1), (3,1), (3,3)$ and $(1,3)$. Find the vertices of the image of S under the transformation given by the matrix $\boldsymbol{M} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Sketch S and the image of S on a coordinate grid.

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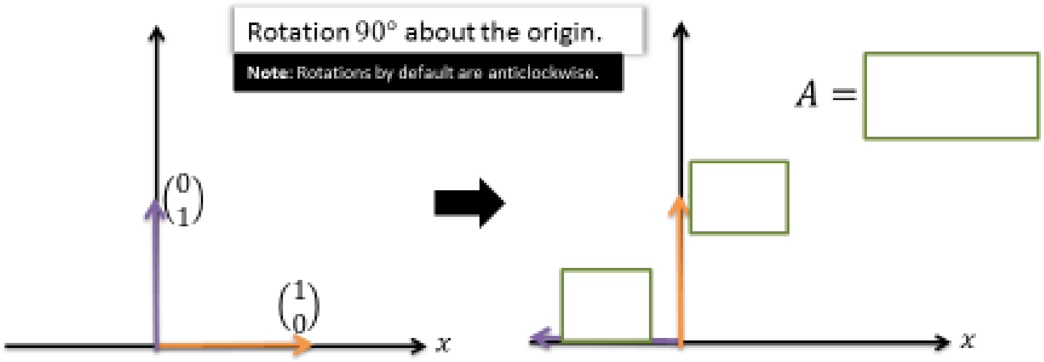
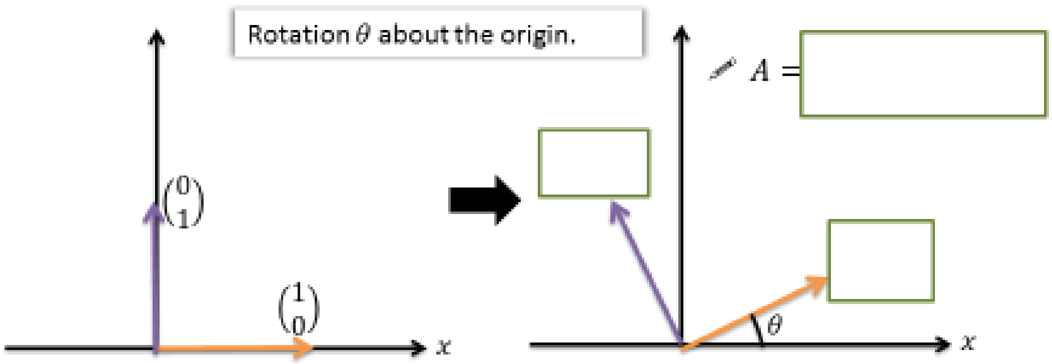
Determining a matrix for a transformation

Rotations





Reflections

Reflection in the x-axis leaves the point (1, 0) unchanged but maps the point (0, 1) to the point (0, -1).

So the matrix representing this transformation is

Reflection in the y-axis maps the point (1, 0) to the point (-1, 0) but leaves the point (0, 1) unchanged.

So the matrix representing this transformation is

Reflection in the line $y = x$ maps the point (1, 0) to the point (0, 1) and maps the point (0, 1) to the point (1, 0).

So the matrix representing this transformation is

Reflection in the line $y = -x$ maps the point (1, 0) to the point (0, -1) and maps the point (0, 1) to the point (-1, 0).

So the matrix representing this transformation is

Warning!! Often mistaken for a rotation!

Test Your Understanding

1. Find the matrix representing a reflection in the line $y = x$.

2. Find the matrix representing a rotation by 270°.

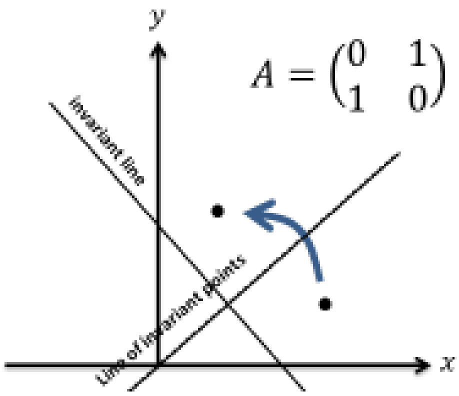
3.

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

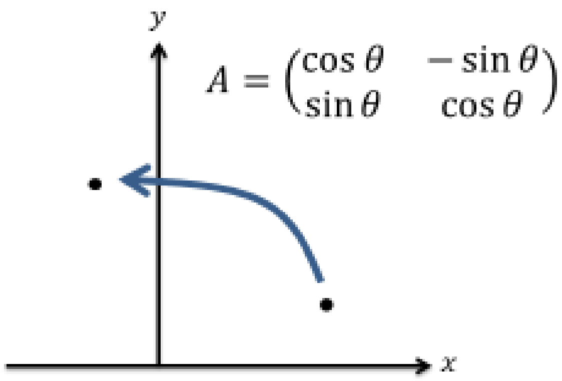
(*a*) Describe fully the transformations described by matrix **C**.

Invariant Lines and Points

Reflection in $y = x$



Rotation



Finding Invariant Points

Finding Invariant Lines

Example

Find the line of invariant points and invariant lines of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

Activity

If possible, fill in either a matrix or a type of transformation (such as reflection or enlargement) that satisfies the conditions for each cell in the grid. If any are not possible, explain why.

		Invariant points	
		Only the origin	Line of invariant points
Invariant lines	none		
	finite number		
	infinite number		

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Enlargements

Enlargements and Invariance

Using the Determinant

Example

$A(1,1), B(1,2), C(2,2)$ are points on a triangle. The transformation with matrix $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ is applied to the triangle to produce a new triangle with vertices A', B' and C'.

(a) Determine the coordinates of $A',\ B', C'$.

(b) What is the area of triangle ABC?

(c) What is the area of triangle $A'B'C'$?

(d) Determine $\det(M)$. What do you notice?

Area of Object	**Transformation Matrix**	**Area of Image**
4	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
3	$\begin{pmatrix} 2 & 0 \\ 9 & 4 \end{pmatrix}$	
9	$\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$	
1	$\begin{pmatrix} -5 & 2 \\ -4 & -2 \end{pmatrix}$	

Test Your Understanding

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(*a*) Find det **A**.

(1)

The triangle R is transformed to the triangle S by the matrix **A**.
Given that the area of triangle S is 72 square units,

(*c*) find the area of triangle R.

(2)

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Combined Transformations

Examples

1. Represent as a single matrix the transformation representing a reflection in the line $y = x$ followed by a stretch on the x axis by a factor of 4.

2. Represent as a single matrix the transformation representing a rotation $90°$ anticlockwise about the point $(0,0)$ followed by a reflection in the line $y = x$.

Test Your Understanding

The transformation U, represented by the 2×2 matrix **P**, is a rotation through 90° anticlockwise about the origin.

(a) Write down the matrix **P**. **(1)**

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line $y = -x$.

(b) Write down the matrix **Q**. **(1)**

Given that U followed by V is transformation T, which is represented by the matrix **R**,

(c) express **R** in terms of **P** and **Q**, **(1)**

(d) find the matrix **R**, **(2)**

(e) give a full geometrical description of T as a single transformation. **(2)**

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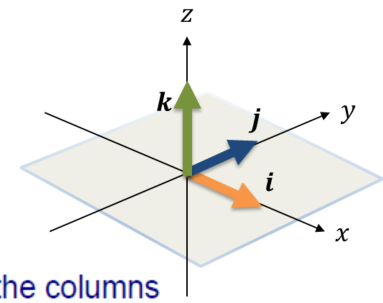
Linear Transformations in 3D

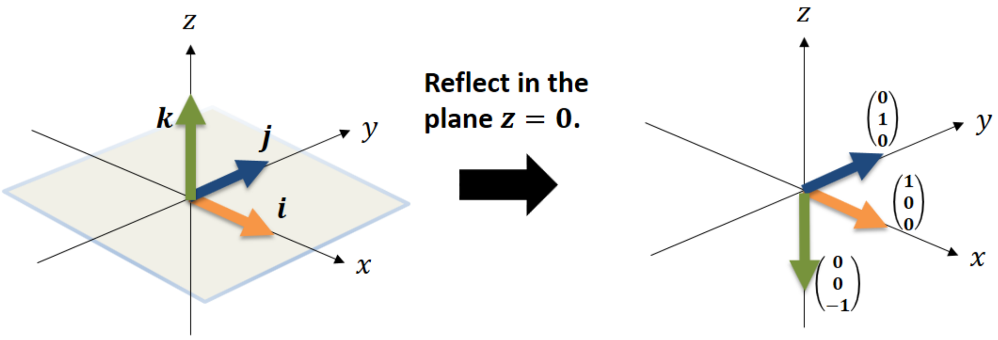


For a transformation in three dimensions, represented by a 3 × 3 matrix, the columns of the matrix represent the images of the point (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

In most of the simple transformations in three dimensions that you will meet, you will see that at least one of the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) maps to itself. One way to identify the transformation is to ignore the row and column for this point, and look at the remaining 2 × 2 matrix. Identify the transformation, and then express it in terms of a three dimensional transformation.

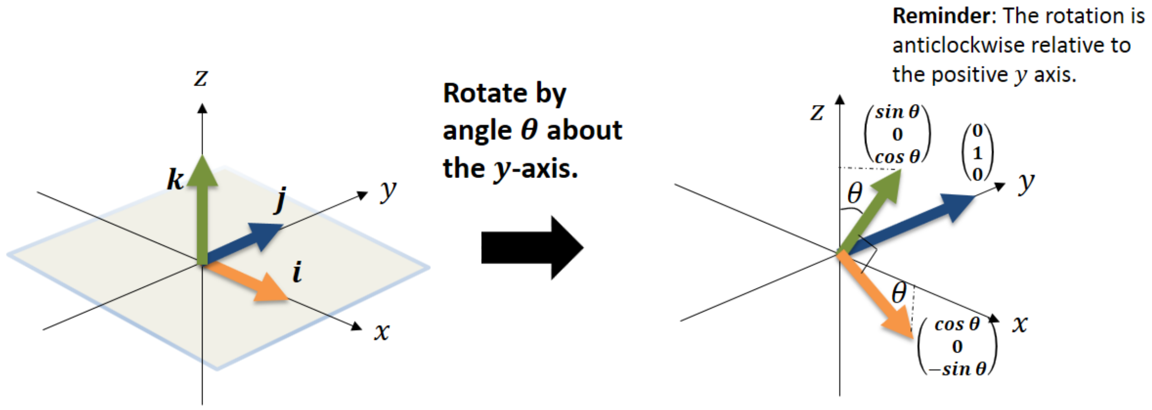
The list below explains how to recognise each of the different types of three-dimensional transformation that you might meet.

Reflections



$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}$$

Rotations



Reminder: The rotation is anticlockwise relative to the positive y axis.
Rotate by angle θ about the y-axis.
z
$\boldsymbol{k}$
$\boldsymbol{j}$
y
$\boldsymbol{i}$
x
z
$\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
y
θ
θ
x
$\begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix}$

Test Your Understanding

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe the transformation represented by $\mathbf{M}$.

(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by $\mathbf{M}$.

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Inverse Matrices for Inverse Transformations

Example

1. Suppose we want to find the inverse of AB, where A and B are non-singular matrices. This means we need to find a matrix X such that X(AB) = I

2. The triangle T has vertices at A, B and C. The matrix $M = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$ transforms T to the triangle T' with vertices at $A'(4,3)$, $B'(4,10)$ and $C'(-4,-3)$. Determine the coordinates of A, B and C.

Test Your Understanding

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}.$$

(*a*) Find det **M**. **(1)**

Given that
$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(*e*) describe fully the single geometrical transformation represented by **A**. **(2)**

The transformation represented by **A** followed by the transformation represented by **B** is equivalent to the transformation represented by **M**.

(*f*) Find **B**. **(4)**

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