7A Introduction to Linear Transformations with Matrices

Linear Transformations

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1. The three transformations **S**, **T** and **U** are defined below. Find the image of the point (2,3) under each of these transformations. State whether each is a *linear* transformation.

$$\boldsymbol{S}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+4 \\ y-1 \end{bmatrix}$$

$$\boldsymbol{T}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2x-y \\ x+y \end{bmatrix}$$

$$\boldsymbol{U}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2y \\ -x^2 \end{bmatrix}$$

Matrices can be used to represent linear transformations:

2. Find matrices to represent these linear transformations:

$$\boldsymbol{T}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2y + x \\ 3x \end{bmatrix}$$

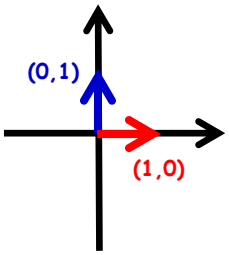
$$\boldsymbol{V}: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} -2y \\ 3x + y \end{bmatrix}$$

3. The square S has coordinates (1,1), (3,1), (3,3) and (1,3).

   Find the coordinates of the vertices of the image of S after the transformation given by the matrix:

$$\boldsymbol{M} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

7B Reflections & Rotations



1. Describe fully the geometrical transformation represented by the matrix:

a)

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b)

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

2. Find a matrix to represent the transformation:

a) ‘Reflection in the y-axis’

b) ‘Enlargement, centre (0,0), scale factor 2’

c) ‘Rotation of 45° anticlockwise about (0,0)’

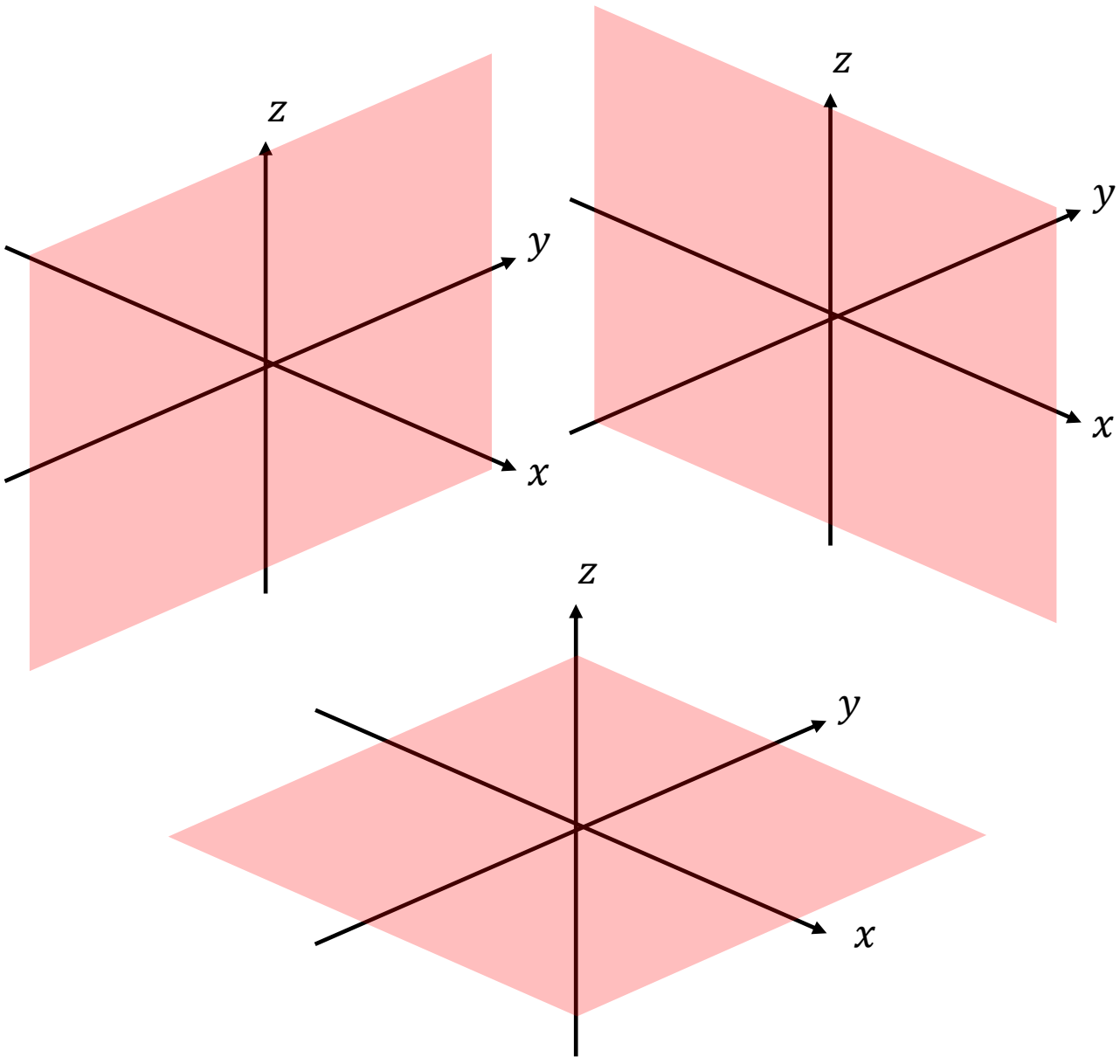
As a general rule, the matrix representing a rotation of angle θ anticlockwise about the origin is:

Final notes:

Invariant point

Invariant Lines

7E 3D Transformations



1. A transformation U, in three dimensions, represents a reflection in the plane $z = 0$.

a) Write down the 3×3 matrix that represents this transformation.

b) Find the image of the point $(-1,2,3)$ under this transformation

Reflection in the yz plane ($x = 0$)

Reflection in the xz plane ($y = 0$)

Reflection in the xy plane ($z = 0$)

2. A transformation U, in three dimensions, represents a 90˚ anticlockwise rotation around the x-axis

a) Write down the 3×3 matrix that represents this transformation.

b) Find the image of the point $(-1,2,3)$ under this transformation

Rotation <u>anticlockwise</u> θ around the x-axis

Rotation <u>anticlockwise</u> θ around the y-axis

Rotation <u>anticlockwise</u> θ around the z-axis

3. The matrix $\boldsymbol{M} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$.

a) Describe the transformation represented by $\boldsymbol{M}$.

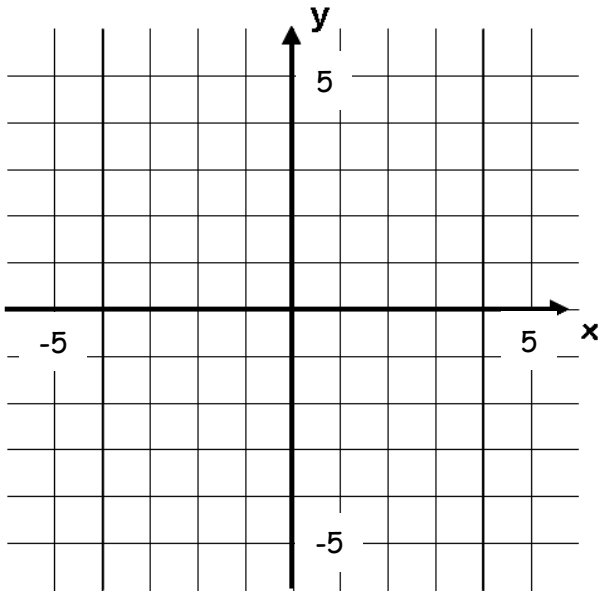
b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by $\boldsymbol{M}$.

7C Enlargements

1. The matrix $\boldsymbol{M} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

a) Find the image T' of the triangle T with vertices (1,1), (1,2) and (2,2) under the transformation represented by $\boldsymbol{M}$.

b) Sketch T and T' on the same set of coordinate axes.



c) Describe geometrically the transformation represented by $\boldsymbol{M}$.

Key note:

The determinant and scale factors:

2. The Matrix $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

a) Describe fully the transformation represented by matrix $\boldsymbol{M}$

b) A triangle T has vertices at (1,0), (4,0) and (4,2). Find the area of the triangle

c) Triangle T is transformed by using matrix $\boldsymbol{M}$. Find the area of the image of T.

7D Multiple Transformations

1. The points A(1,0), B(0,1) and C(2,0) are the vertices of a triangle T. The triangle T is rotated 90° anticlockwise around (0,0) and then the image T' is reflected in the line y = x to obtain the triangle T''.

a) On separate diagrams, draw T, T' and T''

b) i) Find the matrix **P** such that **P**(T) = T'

ii) Find the matrix **Q** such that **Q**(T') = T''

c) By finding a matrix product, find the single matrix that will perform a 90° anticlockwise rotation followed by a reflection in y = x

2. The following matrices represent three different transformations:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad \boldsymbol{Q} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{R} = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}$$

Find the matrix representing the transformation represented by **R**, followed by **Q**, followed by **P** and give a geometrical interpretation of this transformation.

3. $\boldsymbol{M} = \begin{bmatrix} -2\sqrt{2} & -2\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} \end{bmatrix}$

The matrix $\boldsymbol{M}$ represents an enlargement with scale factor k followed by an anticlockwise rotation through angle θ about the origin.

a) Find the value of k

b) Find the value of θ

7F Inverse Matrices & Transformations

1. The triangle T has vertices at A, B and C. The matrix:

$$\boldsymbol{M} = \begin{bmatrix} 4 & -1 \\ 3 & 1 \end{bmatrix}$$

transforms T to the triangle T' with vertices at (4,3), (4,10) and (-4,-3).

Find the coordinates of the points A, B and C

2. The matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ -2 & -5 \end{bmatrix}$ represents a transformation T. Given that T maps point P with coordinates (x,y) onto the point P' with coordinates (6,10):

a) Find the coordinates of P

The matrix $\boldsymbol{B}$ represents a transformation U. Given that the transformation T followed by the transformation U is equivalent to a reflection in the line $y = x$:

b) Find matrix $\boldsymbol{B}$.