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WORKBOOK

ALGEBRAIC PROOF OF ARITHMETIC RESULTS

INTRODUCTION

Did you know that:

- if you add any odd number to any even number the result will always be odd;
- if you multiply two consecutive integers the result will always be even;
- the square of an odd number will always be odd;
- when the product of two consecutive positive integers is added to the larger integer, the result is always a square number!

Also, can you see that:

- for integer values of n greater than zero, the expression $8(n + 7) – 5(n + 7)$ is always a multiple of 3

You may have known some or all of the 'facts' above. If you did not know them, then it would be easy to do a few calculations to convince yourself that the statements above are indeed true.

If we consider the third example – the square of an odd number will always be odd – we can test this by squaring odd numbers:

$1^2 = 1$

$3^2 = 9$

$5^2 = 25$

The answers that I am getting are all odd and if I try a few more examples (7^2, 9^2 etc.) then I still get results which are odd numbers.

This does not, however, prove that squaring an odd number will always produce an odd result, it just shows that it is true for the numbers that I have used in my test – it might be possible that if very large numbers are squared the rule does not work.

The only way to be certain that a rule works 100% of the time is to devise some sort of mathematical proof.

There are many different ways of proving (or disproving) results, but in this workbook we are going to be learning how to 'construct' an algebraic proof.

Note: I will using the letters n and m to represent the numbers (integers) in my proofs.

ODD OR EVEN

Even numbers

Even numbers have a factor of two. This means:

If an expression can be factorised into $2n$ or $2(\ldots)$ it must be even.

Odd numbers

Odd numbers are one more (or one less) than an even number. This means:

If an expression can be written as $2n \pm 1$ or $2(\ldots) \pm 1$ it must be odd.

The following examples use algebra to prove some arithmetic 'facts' about odd and even numbers.

Proof 1

Prove algebraically that the sum of any two even numbers is always even.

Solution:

Let the first even number be $2n$ and the second even number be $2m$

Adding gives:

$$2n + 2m = 2(n + m)$$

This shows that the expression has a factor of 2 so it is even.

Proof 2

Prove algebraically that the sum of any even number and any odd number is always odd.

Solution:

Let the even number be $2n$ and the odd number be $2m + 1$

Adding gives:

$$\begin{aligned} 2n + (2m + 1) &= 2n + 2m + 1 \\ &= 2(n + m) + 1 \end{aligned}$$

Thus, the expression can be written as one more than a multiple of 2, so it is odd

Proof 3

Determine algebraically whether the product of an even number and an odd number is always odd, always even, or could be either odd or even.

Solution:

Let the even number be $2n$ and the odd number be $2m + 1$.

Multiplying gives:

$$2n(2m + 1)$$

This shows a factor of 2 so the result is always even.

Proof 4

Determine algebraically whether the square of an odd number is always odd, always even, or could be either odd or even.

Solution:

Let the odd number be $2n + 1$

Squaring gives:

$$\begin{aligned}(2n+1)^2 &= (2n+1)(2n+1)\\ &= 4n^2+4n+1\\ &= 2(2n^2+2n)+1\end{aligned}$$

The expression can be written as one more than a multiple of 2, so it is always odd.

Proof 5

Prove algebraically that $(2n + 1)^2 - (2n + 1)$ is always an even number for all positive integer values of n.

Solution:

$$\begin{aligned}(2n+1)^2-(2n+1) &= (2n+1)(2n+1)-(2n+1)\\ &= 4n^2+4n+1-2n-1\\ &= 4n^2+2n\\ &= 2(2n^2+n)\end{aligned}$$

This shows a factor of 2 so the expression is even.

SAMPLE QUESTIONS 1

(a) Prove algebraically that the sum of two odd numbers is always even.

(b) Prove algebraically that the product of two odd numbers is always odd.

(c) Prove algebraically that the difference between two odd numbers is always even.

(d) Prove algebraically that the difference between an odd number and an even number is always odd.

(e) Prove algebraically algebraically that the square of an even number is always even.

(f) Prove algebraically that $7(n + 2) - 3(n + 4)$ is always even

(g) Prove algebraically that $n^2 - 2 - (n - 2)^2$ is always an even number.

(h) Prove algebraically that $(n + 1)^2 - (n - 1)^2 + 1$ is odd for all positive integer values of n.

(i) Prove algebraically that $(n + 3)(2n + 1) + (n - 2)(2n + 1)$ is **not** an even number

CONSECUTIVE INTEGERS

If the starting integer is n, then the next integer will be $n + 1$ and the one after it $n + 2$, and so on. Thus a list of consecutive integers would be written as:

$$n, n+1, n+2, n+3, n+4, \ldots$$

Consecutive even numbers

An even number is a multiple of two so I will choose $2n$ to be the starting even number. This means that the list of consecutive **even** numbers would be written as:

$$2n, 2n+2, 2n+4, 2n+6, \ldots$$

Consecutive odd numbers

An odd number is one more (or one less) than a multiple of two so I will choose $2n + 1$ as my starting odd number.

Thus a list of consecutive **odd** numbers would be written as:

$$2n+1, 2n+3, 2n+5, 2n+7, \ldots$$

The following examples use algebra to prove some arithmetic 'facts' about consecutive integers.

Proof 6

Prove algebraically that the sum of any two consecutive integers is always an odd number.

Solution:

Let the starting integer be n.

This means that the next integer will be $n + 1$ and their sum will be:

$$n + (n+1) = 2n + 1$$

This is one more than a multiple of 2 so the result is odd.

Proof 7

Prove algebraically that the sum of any two consecutive even numbers is always an even number.

Solution:

Let the starting even number be $2n$.

This means that the next even number will be $2n + 2$ and their sum will be:

$$\begin{aligned} 2n+(2n+2) &= 4n+2 \\ &= 2(2n+1) \end{aligned}$$

This shows a factor of 2 so the result is even.

Proof 8

Prove algebraically that the product of any two consecutive odd numbers is always odd.

Solution:

Let the first odd number be $2n + 1$.

This means that the next odd number will be $2n + 3$ and their product will be

$$\begin{aligned}(2n+1)(2n+3) &= 4n^2+8n+3\\ &= (4n^2+8n+2)+1\\ &= 2(2n^2+4n+1)+1\end{aligned}$$

This is one more than a multiple of 2, so it is odd.

Proof 9

Prove algebraically that the sum of the squares of any two consecutive integers is always an odd number.

Solution:

Let the first integer be n.

This means that the next integer will be $n + 1$ and the sum of the squares will be

$$\begin{aligned}n^2+(n+1)^2 &= n^2+n^2+2n+1\\ &= 2n^2+2n+1\\ &= 2(n^2+n)+1\end{aligned}$$

This is one more than a multiple of 2, so it is odd.

SAMPLE QUESTIONS 2

(a) Prove algebraically that the sum of any two consecutive odd numbers is always even.

(b) Prove algebraically that the difference between any two consecutive odd numbers is always two.

(c) Prove algebraically that the product of any two consecutive even numbers is always even.

(d) Prove algebraically that the mean of three consecutive integers is always the middle number

(e) Prove algebraically that the sum of any three consecutive even integers is always even.

(f) Prove algebraically that the difference between the squares of any two consecutive integers is always an odd number.

MULTIPLES

If an expression can be written as $3n$ or $3(\dots)$ then it must be a multiple of 3.

The same applies for 4, 5, 6 etc.

In general, if an expression can be written as kn or $k(\dots)$ then it must be a multiple of k.

Example 10

Prove algebraically that the sum of any three consecutive integers is always a multiple of 3.

Solution:

Let the first integer be n

This means that the next two integers will be $n + 1$ and $n + 2$, and their sum will be

$$\begin{aligned} n+(n+1)+(n+2) &= 3n+3 \\ &= 3(n+1) \end{aligned}$$

This shows a factor of 3 so the expression is a multiple of 3

Example 11

Prove algebraically that $3(n + 4) + 5(n + 6)$ is always a multiple of 4

Solution:

$$\begin{aligned} 3(n+4)+5(n+6) &= 3n+12+5n+30 \\ &= 8n+32 \\ &= 4(2n+8) \end{aligned}$$

This shows a factor of 4 so the expression is a multiple of 4

Example 12

Prove that the square of any odd number is always one more than a multiple of 4

Solution:

If the odd number is $2n + 1$ then squaring it gives:

$$\begin{aligned} (2n+1)^2 &= (2n+1)(2n+1) \\ &= 4n^2+4n+1 \\ &= 4(n^2+n)+1 \end{aligned}$$

Which is one more than a multiple of 4

SAMPLE QUESTIONS 3

(a) Prove algebraically that $5(n + 7) + 3(n + 3)$ is always a multiple of 4

(b) Prove algebraically that $8(n + 7) – 5(n + 4)$ is always a multiple of 3

(c) Prove algebraically that $7(n + 8) + 5(n – 4)$ is always a multiple of 12

(d) Prove algebraically that $(m + 2)^2 – m^2 – 12$ is always a multiple of 4

(e) Prove algebraically that $(n + 6)^2 – (n + 2)^2$ is always a multiple of 8, for all positive integer values of n.

(f) Prove algebraically that $(5n – 3)^2 – 3(3 – 10n)$ is always a multiple of 5

(g) Prove algebraically that $(4n + 2)^2 – 12(n + 1)$ is always a multiple of 4

(h) Prove algebraically that $(4n + 1)^2 – (4n – 1)^2$ is always a multiple of 8, for all positive integer values of n.

(i) Show that when x is a whole number $7(2x + 1) + 6(x+ 3)$ is always a multiple of 5

(j) Prove algebraically that the sum of three consecutive odd numbers is always a multiple of 3

(k) Prove that $(2n + 1)^2 – (2n – 1)^2 – 2$ is not a multiple of 4 for all positive integer values of n.

MIXED QUESTIONS

(a) Prove algebraically that the sum of the squares of any two odd positive integers is always even.

(b) Prove that the sum of the squares of two consecutive odd numbers is always 2 more than a multiple of 8.

(c) Prove algebraically that the sum of any two consecutive odd numbers is always a multiple of 4.

(d) Prove that the sum of the squares of any three consecutive odd numbers is always 11 more than a multiple of 12.

(e) Prove algebraically that $n^2 - 2 - (n - 2)^2$ is always an even number.

(f) Prove that $(3n + 1)^2 – (3n – 1)^2$ is always a multiple of 12, for all positive integer values of n.

(g) Prove that $(8n + 2)^2 – (8n – 3)^2$ is always a multiple of 5

(h) Prove algebraically that the sum of four consecutive integers is not divisible by 4.

(i) Prove that $(n – 1)^2 + n^2 + (n + 1)^2 = 3n^2 + 2$

EXTENSION 1

(a) Prove that an odd number cubed is also odd.

(b) Prove that the sum of two consecutive multiples of 5 is always an odd number.

(c) Given that $2(x - n) = x + 5$ where n is an integer, prove that x must be an odd number.

(d) Given that $4(x + n) = 3x + 10$ where n is an integer, prove that x must be an even number.

(e) Prove that if the difference of two numbers is 4, then the difference of their squares is a multiple of 8

(f) Prove that for any three consecutive numbers, the difference between the squares of the first and last numbers is 4 times the middle number.

(g) If n is a positive integer greater than 1, prove that $n^3 - n$ is a multiple of 6 for all possible values of n.

(h) Prove that the difference between the squares of any two consecutive integers is equal to the sum of the two integers.

(i) Prove that the sum of the squares of two consecutive odd numbers is always 2 more than a multiple of 8

(j) The product of two consecutive positive integers is added to the larger of the two integers. Prove that the result is always a square number.

EXTENSION 2

(a) If a, b and c are three consecutive numbers, prove that $c^2 - a^2 = 4b$

(b) Given that n is an integer, prove algebraically that the sum of $(n + 2)(n + 1)$ and $n + 2$ is always a square number.

(c) Prove that for any two numbers the product of their difference and their sum is equal to the difference of their squares.

(d) If a, b, c and d are four consecutive integers, show that the product of the first and last is two less than the product of the second and third.

(e) Given that n is an integer, prove that $(n - 2)(n + 3) + (6 - n)$ is a square number.

(f) Given that triangle numbers can be represented by $T_n = \dfrac{n(n+1)}{2}$.

Prove that eight times **<u>any</u>** triangle number is one less than a square number.

(g) n is an integer.

Prove algebraically that the sum of $\frac{1}{2}n(n+1)$ and $\frac{1}{2}(n+1)(n+2)$ is always a square number.

(h) x is a positive whole number.
Explain why the expression $2x^2 + 5x + 2$ can never have a value that is a prime number.

(i) $2^{61} - 1$ is a prime number. Explain why $2^{61} + 1$ must be a multiple of 3.