Lower 6 Chapter 7

Algebraic Methods

Chapter Overview

1. Algebraic Fractions
2. Algebraic Long Division
3. Factor Theorem
4. Proof

2 **Algebra and functions** *continued*	2.6	**Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.** Simplify rational expressions, including by factorising and cancelling, and algebraic division (by linear expressions only).	**Only division by $(ax + b)$ or $(ax - b)$ will be required. Students should know that if $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$.** **Students may be required to factorise cubic expressions such as $x^3 + 3x^2 - 4$ and $6x^3 + 11x^2 - x - 6$.** Denominators of rational expressions will be linear or quadratic, e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$
1 **Proof**	1.1	**Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including:**	**Examples of proofs:**
		Proof by deduction	**Proof by deduction** **e.g. using completion of the square, prove that $n^2 - 6n + 10$ is positive for all values of n or, for example, differentiation from first principles for small positive integer powers of x** or proving results for arithmetic and geometric series. **This is the most commonly used method of proof throughout this specification**
		Proof by exhaustion	**Proof by exhaustion** **This involves trying all the options. Suppose x and y are odd integers less than 7. Prove that their sum is divisible by 2.**
		Disproof by counter example	**Disproof by counter example** **e.g. show that the statement "$n^2 - n + 1$ is a prime number for all values of n" is untrue**

Simplifying Algebraic Fractions

Examples

1. $\frac{x^2-1}{x^2+x}$

2. $\frac{x^2+3x+2}{x+1}$

3. $\frac{2x^2+11x+12}{x^2+9x+20}$

4. $\frac{4-x^2}{x^2+2x-8}$

Exercise 7A Page 138

Algebraic Long Division

Examples

1.

2.

Test your understanding

1. Find the remainder when $2x^3 - 5x^2 - 16x + 10$ is divided by $x - 4$.

2. Divide $8x^3 - 1$ by $2x - 1$.

Exercise 7B Page 141

The Factor Theorem

Examples

1. Show that $(x-2)$ is a factor of x^3+x^2-4x-4.

2. Fully factorise $2x^3+x^2-18x-9$.

Using the factor theorem to find unknown coefficients:

1. Given that $2x + 1$ is a factor of $6x^3 + ax^2 + 1$, determine the value of a.

Test your understanding

$$f(x) = 6x^3 + 13x^2 - 4$$

(a) ~~Use the remainder theorem to find the remainder when $f(x)$ is divided by $(2x + 3)$~~. **(2)**

(b) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2)**

(c) Factorise $f(x)$ completely. **(4)**

2. Given that $3x - 1$ is a factor of $3x^3 + 11x^2 + ax + 1$, determine the value of a.

Extension

1. *[MAT 2006 1E]* The cubic $x^3 + ax + b$ has both $(x - 1)$ and $(x - 2)$ has factors. Determine the values of a and b.

2. [MAT 2009 1I] The polynomial $n^2x^{2n+3} - 25nx^{n+1} + 150x^7$ has $x^2 - 1$ as a factor

A) for no values of n;

B) for $n = 10$ only;

C) for $n = 15$ only;

D) for $n = 10$ and $n = 15$ only.

The **remainder theorem** states that if $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$. This similarly works whenever a makes the divisor 0.

(No longer required for A Level)

3. [MAT 2013 1G] Let $n \geq 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n)$$

What is the remainder, in terms of n, when $p_n(x)$ is divided by $p_{n-1}(x)$?

Exercise 7C Page 145

Proof

- A **conjecture** is a mathematical statement that has yet to be proven.
- A **theorem** is a mathematical statement that has been proven.

Proof by Deduction:

Examples:

1. **"Prove that the product of two odd numbers is odd."**

2. **"Prove that $(3x+2)(x-5)(x+7) \equiv 3x^3 + 8x^2 - 101x - 70$"**

3. **Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5**

Test your Understanding:

Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8.

Extension

[STEP I 2005 Q1] 47231 is a five-digit number whose digits sum to

$4 + 7 + 2 + 3 + 1 = 17.$

(i) Prove that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

Exercise 7D Page 149

Proof by Exhaustion

Example: **Prove that $n^2 + n$ is even for all integers n.**

Disproof by counter-example

Example: **Disprove the statement:**

"$n^2 - n + 41$ is prime for all integers n."

[Proof by contradiction covered in Year 2]

Exercise 7E Page 152