Chapter 4

# Graphs and Transformations

## Chapter Overview

1. Polynomial Graphs
    a. Cubic Graphs
    b. Quartic Graphs
    c. Reciprocal Graphs
2. Points of Intersection
3. Graph Transformations

| | |
|---|---|
| **Understand and use graphs of functions; sketch curves defined by simple equations including polynomials** | **Graph to include simple cubic and quartic functions,**<br>**e.g. sketch the graph with equation** $y = x^2(2x-1)^2$ |
| $y = \dfrac{a}{x}$ **and** $y = \dfrac{a}{x^2}$<br>**(including their vertical and horizontal asymptotes)**<br>**Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.** | **The asymptotes will be parallel to the axes e.g. the asymptotes of the curve with equation** $y = \dfrac{2}{x+a} + b$ **are the lines with equations** $y = b$ **and** $x = -a$ |

| | |
|---|---|
| **Understand the effect of simple transformations on the graph of $y = f(x)$, including sketching associated graphs:**<br><br>$y = af(x)$, $y = f(x) + a$,<br><br>$y = f(x + a)$, $y = f(ax)$<br><br>and combinations of these transformations | Students should be able to find the graphs of $y = \lvert f(x) \rvert$ and $y = \lvert f(-x) \rvert$, given the graph of $y = f(x)$.<br><br>Students should be able to apply a combination of these transformations to any of the functions in the A Level specification (quadratics, cubics, quartics, reciprocal, $\frac{a}{x^2}$, $\lvert x \rvert$, $\sin x$, $\cos x$, $\tan x$, $e^x$ and $a^x$) and sketch the resulting graph.<br><br>Given the graph of $y = f(x)$, students should be able to sketch the graph of, e.g. $y = 2f(3x)$, or $y = f(-x) + 1$,<br><br>and should be able to sketch (for example)<br><br>$y = 3 + \sin 2x$, $y = -\cos\left(x + \frac{\pi}{4}\right)$ |

# Polynomial Graphs

| Equation | If $a > 0$ | Resulting Shape | If $a < 0$ | Resulting Shape |
|---|---|---|---|---|
| $y = ax^2 + bx + c$ | As $x \to \infty,\ y \to \infty$<br>As $x \to -\infty,\ y \to \infty$ | | As $x \to \infty,\ y \to -\infty$<br>As $x \to -\infty,\ y \to -\infty$ | |
| $y = ax^3 + bx^2$<br>$+cx + d$ | | | | |
| $y = ax^4 + bx^3 +$<br>$cx^2 + dx + e$ | | | | |
| $y = ax^5 + bx^4 + \cdots$ | | | | |

# Cubics

Examples

1. Sketch the curve with equation $y = (x - 2)(1 - x)(1 + x)$

We consider the shape, the roots and the y – intercept.

2. Sketch the curve with equation $y = x^2(x-1)$

3. Sketch the curve with equation $y = (2-x)(x+1)^2$

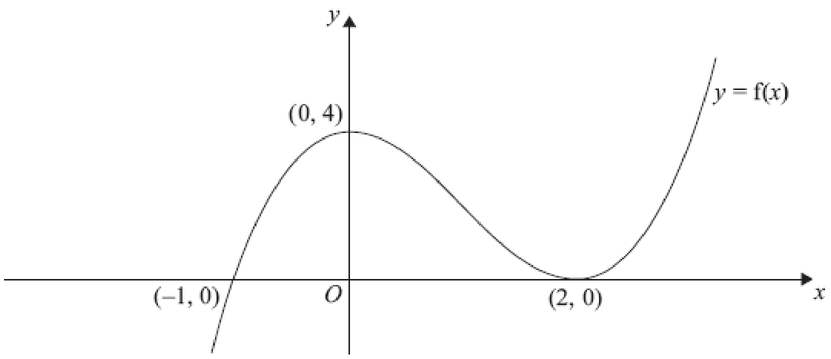
4. Sketch the curve with equation $y = (x-4)^3$

5. Sketch the curve with equation $y = (x + 1)(x^2 + x + 1)$

<u>Finding the equation: example</u>

The graph shows a sketch of the curve $C$ with equation $y = f(x)$. The curve $C$ passes through the point (–1, 0) and touches the $x$-axis at the point (2, 0). The curve $C$ has a maximum at the point (0, 4). The equation of the curve $C$ can be written in the form $y = x^3 + ax^2 + bx + c$ where $a$, $b$ and $c$ are integers.

Calculate the values of $a, b, c$.

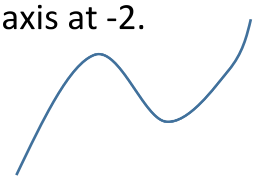


Test Your Understanding:

1. Sketch the curve with equation $y = x(x-3)^2$

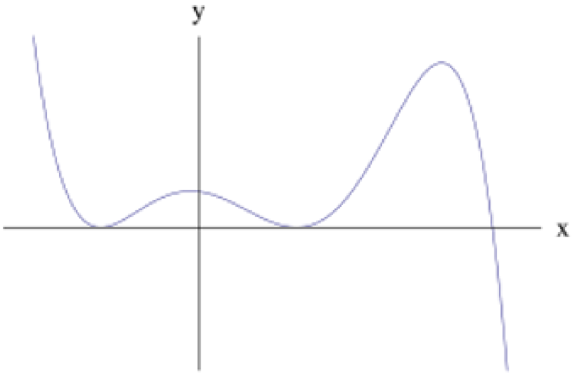
2. Sketch the curve with equation $y = -(x+2)^3$

3. A curve has this shape , touches the $x$ axis at 3 and crosses the $x$ axis at -2. Give a suitable equation for this graph.



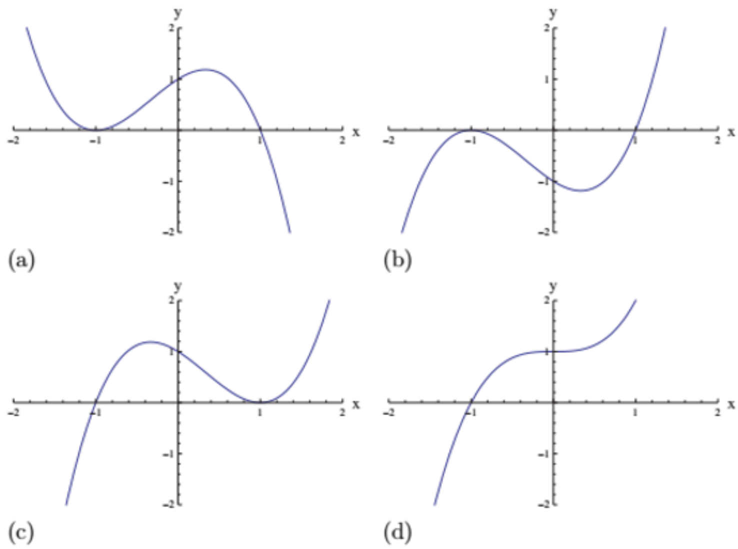
4. Extension. Sketch the curve with equation $y = 2x^2(x-1)(x+1)^3$

[MAT 2012 1E] Which one of the following equations could possibly have the graph given below?




A) $y = (3-x)^2(3+x)^2(1-x)$
B) $y = -x^2(x-9)(x^2-3)$
C) $y = (x-6)(x-2)^2(x+2)^2$
D) $y = (x^2-1)^2(3-x)$

[MAT 2011 1A] A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axis?



# Quartics:

Examples:

1. Sketch the curve with equation $y = x(x+1)(x-2)(x-3)$

2. Sketch the curve with equation $y = (x-2)^2(x+1)(3-x)$

3. Sketch the curve with equation $y = (x+1)(x-1)^3$

4. Sketch the curve with equation $y = (x-2)^4$

Test Your Understanding

1. Sketch the curve with equation $y = x^2(x+1)(x-1)$

2. Sketch the curve with equation $y = -(x+1)(x-3)^3$

Extension:

*[STEP I 2012 Q2a]*

a. Sketch $y = x^4 - 6x^2 + 9$
b. For what values of $b$ does the equation $\mathrm{y} = x^4 - 6x^2 + b$ have the following number of <u>distinct</u> roots (i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 4.

Exercise 4B Page 65

Reciprocal Graphs

1. Sketch $y = \frac{1}{x}$

2. Sketch $y = -\frac{3}{x}$

3. Sketch $y = \frac{3}{x^2}$

4. Sketch $y = -\frac{4}{x^2}$

5. On the same axes, sketch $y = \frac{1}{x}$ and $y = \frac{3}{x}$

Exercise 4C Page 67

## Points of Intersection

If $y = f(x)$ and $y = g(x)$, then the $x$ values of the points of intersection can be found when $f(x) = g(x)$.

Examples:

1. On the same diagram sketch the curves with equations $y = x(x-3)$ and $y = x^2(1-x)$. Find the coordinates of their points of intersection.

2. On the same diagram sketch the curves with equations $y = x^2(3x-a)$ and $y = \frac{b}{x}$, where $a, b$ are positive constants. State, giving a reason, the number of real solutions to the equation $x^2(3x-a) - \frac{b}{x} = 0$

## Test Your Understanding

On the same diagram sketch the curves with equations $y = x(x-4)$ and $y = x(x-2)^2$, and hence find the coordinates of any points of intersection.

## Extension

1. [MAT 2005 1B]

The equation $(x^2+1)^{10} = 2x - x^2 - 2$

A) has $x = 2$ as a solution;

B) has no real solutions;

C) has an odd number of real solutions;

D) has twenty real solutions.

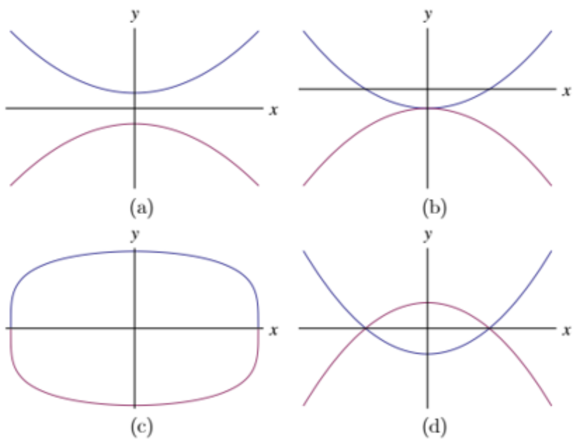
2. [MAT 2010 1A] The values of $k$ for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

A) $k \leq 0$ B) $k \geq -4$

C) $k \geq 0$ or $k \leq -4$ D) $-4 \leq k \leq 0$

3. [MAT 2013 1D]

Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



# Transformations of Graphs

It is important to understand the effects of simple transformations on the graph $y = f(x)$.

$For\ y = f(x)$:

| Function | Effect |
|---|---|
| $f(x + a)$ | |
| $f(x - a)$ | |
| $f(x) + a$ | |
| $f(x) - a$ | |
| $f(ax)$ | |
| $af(x)$ | |
| $f(-x)$ | |
| $-f(x)$ | |

We can think of it like this:

| | Affects which axis? | What we expect or opposite? |
|---|---|---|
| Change **inside** $f(\,)$ | | |
| Change **outside** $f(\,)$ | | |

## Examples: Describe the transformation

1. $y = f(x - 3)$

2. $y = f(x) + 4$

3. $y = f(5x)$

4. $y = 2f(x)$

Example

1. Sketch $y = x^2 + 3$

2. Sketch $y = \frac{2}{x+1}$

3. Sketch $y = x(x+2)$. On the same axes, sketch $y = (x-a)(x-a+2)$, where $a > 2$.

4. Sketch $y = x^2(x-4)$. On the same axes, sketch the graph with equation $y = (2x)^2(2x-4)$.

## Reflections

Example

If $y = x(x + 2)$, sketch $y = f(x)$ and $y = -f(x)$ on the same axes.

Test your understanding

1. If $y = (x + 1)(x - 2)$, sketch $y = f(x)$ and $y = f(\frac{x}{3})$ on the same axes.

2. Sketch the graph of $y = \frac{2}{x} + 1$, ensuring you indicate any intercepts with the axes.

The effect of transformations on specific points

Sometimes you will not be given the original function, but will be given a sketch with specific points and features you need to transform.

Where would each of these points end up?

| $\boldsymbol{y = f(x)}$ | $\boldsymbol{(4, 3)}$ | $\boldsymbol{(1, 0)}$ | $\boldsymbol{(6, -4)}$ |
|---|---|---|---|
| $y = f(x+1)$ | | | |
| $y = f(2x)$ | | | |
| $y = 3f(x)$ | | | |
| $y = f(x) - 1$ | | | |
| $y = f\left(\frac{x}{4}\right)$ | | | |
| $y = f(-x)$ | | | |
| $y = -f(x)$ | | | |

## Test Your Understanding

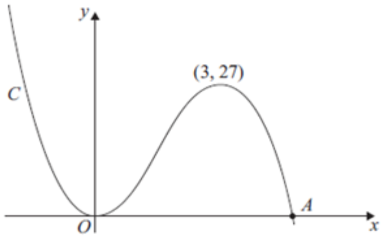



Figure 1 shows a sketch of the curve $C$ with equation $y = f(x)$, where

$$f(x) = x^2(9 - 2x).$$

There is a minimum at the origin, a maximum at the point (3, 27) and $C$ cuts the $x$-axis at the point $A$.

(*a*) Write down the coordinates of the point $A$.

**(1)**

(*b*) On separate diagrams sketch the curve with equation

(i) $y = f(x + 3)$,

(ii) $y = f(3x)$.

On each sketch you should indicate clearly the coordinates of the maximum point and any points where the curves cross or meet the coordinate axes.

**(6)**

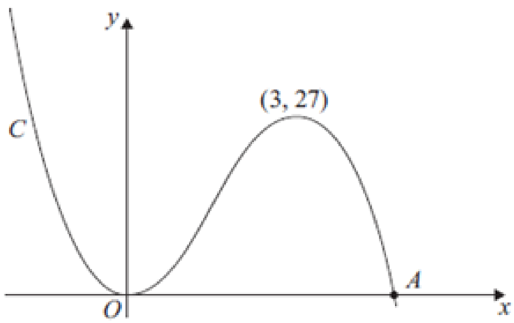
The curve with equation $y = f(x) + k$, where $k$ is a constant, has a maximum point at (3, 10).

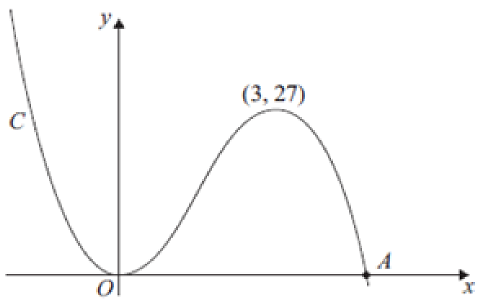
(*c*) Write down the value of $k$.

**(1)**

a)

b)




c)

Exercise 4G Page 80