Chapter 2

Quadratics

Chapter Overview

1. Solving Quadratic Equations
2. Quadratic Functions
3. Quadratic Graphs
4. The Discriminant
5. Modelling with Quadratics

2.3	**Work with quadratic functions and their graphs.**	**The notation $f(x)$ may be used**
	The discriminant of a quadratic function, including the conditions for real and repeated roots.	**Need to know and to use** $b^2 - 4ac > 0$, $b^2 - 4ac = 0$ **and** $b^2 - 4ac < 0$
	Completing the square.	$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$
	Solution of quadratic equations	**Solution of quadratic equations by factorisation, use of the formula, use of a calculator and completing the square.**
	including solving quadratic equations in a function of the unknown.	**These functions could include powers of x, trigonometric functions of x, exponential and logarithmic functions of x.**

Solving Quadratic Equations

Examples

1. $(x-1)^2 = 5$

2. $x^2 + 5x - 6 = 0$

3. Solve $x - 6\sqrt{x} + 8 = 0$

4. $x^2 + 5x - 6 = 0$

Test your understanding

1. $(x+3)^2 = x+5$

2. $(2x+1)^2 = 5$

3. $\sqrt{x+3} = x-3$

4. $2x + \sqrt{x} - 1 = 0$

Extension

(i) Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2} - 22 = 0\,;$

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 = 0\,.$

<u>Solving by Completing the Square</u>

Worked Examples (a = 1):

1. $x^2 + 12x$

2. $x^2 + 8x$

3. . $x^2 - 2x$

4. $x^2 - 6x + 7$

More complicated examples (a not equal to 1):

1. Express $2x^2 + 12x + 7$ in the form $a(x + b)^2 + c$

2. Express $5 - 3x^2 + 6x$ in the form $a - b(x + c)^2$

Test Your Understanding:

1. Express $3x^2 - 18x + 4$ in the form $a(x + b)^2 + c$

2. Express $20x - 5x^2 + 3$ in the form $a - b(x + c)^2$

Solving by Completing the Square:

Note: Previously we factorised out the 3. This is because $3x^2 - 18x + 4$ on its own is an **expression,** so dividing by 3 (instead of factorising) would change the expression. However, in an equation, we can divide both sides by 3 without affecting the solutions.

Example

Solve the equation $3x^2 - 18x + 4 = 0$ by completing the square.

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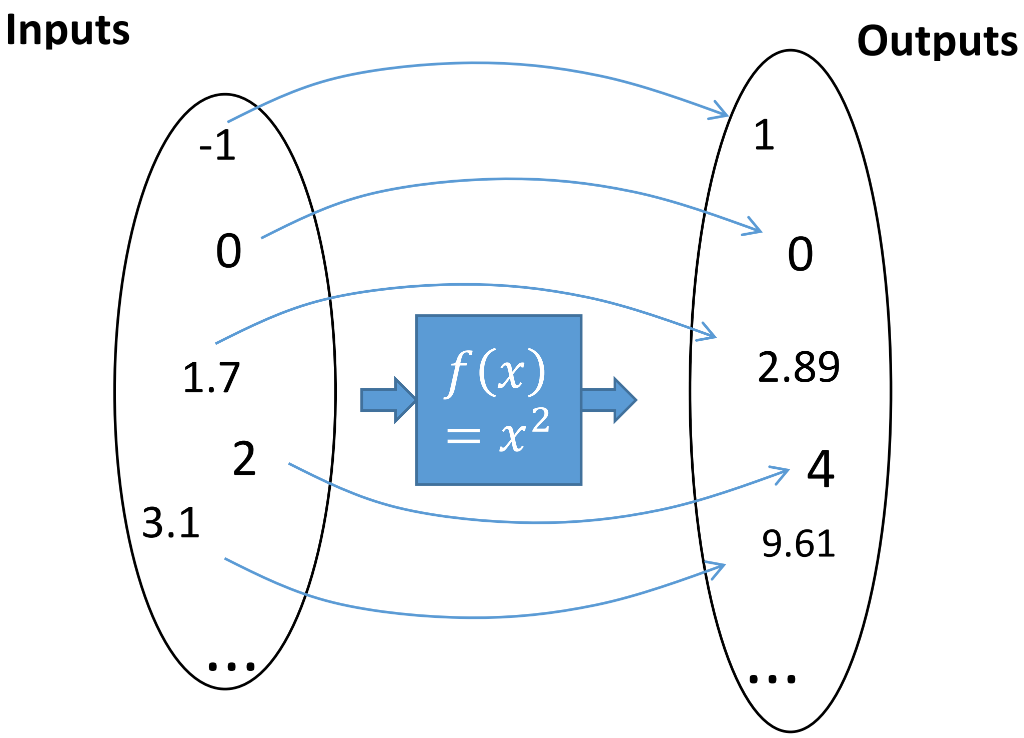
Proving the Quadratic Formula:

Functions:



Examples:

1. If $f(x) = x^2 - 3x$ and $g(x) = x + 5, \quad x \in \mathbb{R}$

   a) Find $f(-4)$

   b) Find the values of x for which $f(x) = g(x)$

   c) Find the roots of $f(x)$.

   d) Find the roots of $g(x)$.

2. Determine the minimum value of the function $f(x) = x^2 - 6x + 2$, and state the value of x for which this minimum occurs.

Test Your Understanding:

$f(x)$	Completed square	Min/max value of $f(x)$	x for which this min/max occurs
$x^2 + 4x + 9$			
$x^2 - 10x + 21$			
$10 - x^2$			
$8 - x^2 + 6x$			

1. Find the minimum value of $f(x) = 2x^2 + 12x - 5$ and state the value of x for which this occurs.

2. Find the roots of the function $f(x) = 2x^2 + 3x + 1$

3. Find the roots of the function $f(x) = x^4 - x^2 - 6$

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Quadratic Graphs:

Example: Sketch the graph of $y = x^2 + 3x - 4$ and find the coordinates of the turning point.

Example: Sketch the graph of $y = 4x - 2x^2 - 3$ and find the coordinates of the turning point. Write down the equation of the line of symmetry.

Test Your Understanding

Sketch the following, indicating any intercepts with the axis, the turning point and the equation of the line of symmetry.

1. $y = x^2 + 4$

2. $y = x^2 - 7x + 10$

3. $y = 5x + 3 - 2x^2$

4. $y = x^2 + 4x + 11$

Extension:

[MAT 2003 1H] Into how many regions is the plane divided when the following three parabolas are drawn?

$$y = x^2$$

$$y = x^2 - 2x$$

$$y = x^2 + 2x + 2$$

The Discriminant

Quickfire questions:

Equation	Discriminant	No. of distinct real roots
$x^2 + 3x + 4 = 0$		
$x^2 - 4x + 1 = 0$		
$x^2 - 4x + 4 = 0$		
$2x^2 - 6x - 3 = 0$		
$x - 4 - 3x^2 = 0$		
$1 - x^2 = 0$		

Example:

8. The equation $x^2 + 2px + (3p + 4) = 0$, where p is a positive constant, has equal roots.

(*a*) Find the value of p.

(4)

(*b*) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$.

(2)

Test Your Understanding:

1. $x^2 + 5kx + (10k + 5) = 0$ where k is a positive constant.

Given that this equation has equal roots, determine the value of k.

2. Find the range of values of k for which $x^2 + 6x + k = 0$ has two distinct real solutions.

Extension:

1. [MAT 2009 1C] Given a real constant c, the equation $x^4 = (x - c)^2$ has four real solutions (including possible repeated roots) for:

   A) $c \leq \frac{1}{4}$
   B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$
   C) $c \leq -\frac{1}{4}$
   D) all values of c

2. [MAT 2006 1B] The equation $(2 + x - x^2)^2 = 16$ has how many real root(s)?

3. [MAT 2011 1B] A rectangle has perimeter P and area A. The values P and A must satisfy:

   A) $P^3 > A$
   B) $A^2 > 2P + 1$
   C) $P^2 \geq 16A$
   D) $PA > A + P$

Modelling

Example

A spear is thrown over level ground from the top of a tower.

The height, in metres, of the spear above the ground after t seconds is modelled by the function: $h(t) = 12.25 + 14.7t - 4.9t^2, \quad t \geq 0$

a) Interpret the meaning of the constant term 12.25 in the model.

b) After how many seconds does the spear hit the ground?

c) Write $h(t)$ in the form $A - B(t - C)^2$, where $A,\ B$ and C are constants to be found.

d) Using your answer to part c or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached?

Test Your Understanding

A rectangular car park has a perimeter of 184 metres, and the diagonal of the car park measures 68 metres.

(i) By labelling the length of the car park as x metres, formulate an equation and check that $x = 32$ satisfies the equation. Hence find the dimensions of the car park.

(ii) Sketch the graph of the quadratic expression in part (i), and interpret each intersection with the x-axis in terms of the car park.

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