Lower 6 Chapter 12

Differentiation

Chapter Overview

1. First Principles and finding the derivative of polynomials.
2. Find equations of tangents and normal to curves.
3. Identify increasing and decreasing functions.
4. Find and understand the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$
5. Find stationary points and determine their nature.
6. Sketch a gradient function.
7. Model real-life problems.

7 **Differentiation**	7.1	**Understand and use the derivative of $\mathrm{f}(x)$ as the gradient of the tangent to the graph of $y = \mathrm{f}(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change** **sketching the gradient function for a given curve** **second derivatives** **differentiation from first principles for small positive integer powers of** x and for $\sin x$ and $\cos x$	**Know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the rate of change of y with respect to x.** **The notation $\mathrm{f}'(x)$ may be used for the first derivative and $\mathrm{f}''(x)$ may be used for the second derivative.** **Given for example the graph of $y = \mathrm{f}(x)$, sketch the graph of $y = \mathrm{f}'(x)$ using given axes and scale. This could relate speed and acceleration for example.** **For example, students should be able to use, for $n = 2$ and $n = 3$, the gradient expression** $\lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$ **Students may use δx or h**

Topics	What students need to learn:		
		Content	**Guidance**
7 **Differentiation** *continued*	7.1 *cont.*	**Understand and use the second derivative as the rate of change of gradient;** connection to convex and concave sections of curves and points of inflection.	Use the condition $\mathrm{f}''(x) > 0$ implies a minimum and $\mathrm{f}''(x) < 0$ implies a maximum for points where $\mathrm{f}'(x) = 0$ Know that at an inflection point $\mathrm{f}''(x)$ changes sign. Consider cases where $\mathrm{f}''(x) = 0$ and $\mathrm{f}'(x) = 0$ where the point may be a minimum, a maximum or a point of inflection (e.g. $y = x^n, n > 2$)
	7.2	**Differentiate x^n, for rational values of n, and related constant multiples, sums and differences.** Differentiate e^{kx} and a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples. Understand and use the derivative of $\ln x$	**For example, the ability to differentiate expressions such as** $(2x + 5)(x - 1)$ **and** $\frac{x^2 + 3x - 5}{4x^{\frac{1}{2}}}, x > 0,$ **is expected.** Knowledge and use of the result $\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx}\ln a$ is expected.
	7.3	**Apply differentiation to find gradients, tangents and normals**	**Use of differentiation to find equations of tangents and normals at specific points on a curve.**
		maxima and minima and stationary points. points of inflection **Identify where functions are increasing or decreasing.**	**To include applications to curve sketching. Maxima and minima problems may be set in the context of a practical problem.** **To include applications to curve sketching.**

The Gradient Function

Example

The point A with coordinates $(4,16)$ lies on the curve with equation $y = x^2$.

At point A the curve has gradient g.

a) Show that $g = \lim_{h \to 0}(8 + h)$

b) Deduce the value of g.

Example

Prove from first principles that the derivative of $x^3 - 2x = 3x^2 - 2$

Test you understanding

Prove **from first principles** that the derivative of x^4 is $4x^3$.

Ex 12A/B page261

Differentiating x^n

Examples

1. $y = x^5$

2. $f(x) = x^{\frac{1}{2}}$

3. $y = 2x^6$

4. $f(x) = \frac{x}{x^4}$

5. $y = \sqrt{x^6}$

Test Your Understanding

1. $y = x^7$

2. $y = 3x^{10}$

3. $f(x) = \frac{x^{\frac{1}{2}}}{x^2}$

4. $y = ax^a$

5. $f(x) = \sqrt{49x^7}$

Differentiating Multiple Terms

Example

Differentiate $y = x^2 + 4x + 3$

Questions

1. $y = 2x^2 - 3x$

2. $y = 4 - 9x^3$

3. $y = 5x + 1$

4. $y = ax$ (a is a constant)

5. $y = 6x - 3 + px^2$ (p is a constant)

<u>Harder Example</u>

Let $f(x) = 4x^2 - 8x + 3$

a) Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8.

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.

Test Your Understanding

Let $f(x) = x^2 - 4x + 2$

a) Find the gradient of $y = f(x)$ at the point $(1, -1)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 5.

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 2 - x$.

Ex 12C/D page265

Differentiating Harder Expressions

1. Turn roots into powers $y = \sqrt{x}$

2. Split up fractions $y = \frac{x^2+3}{\sqrt{x}}$

3. Expand out brackets $y = x^2(x - 3)$

4. Beware of numbers in denominators $y = \frac{1}{3x}$

Test your understanding

Differentiate the following

1. $y = \frac{1}{\sqrt{x}}$

2. $y = \frac{2+x^3}{x^2}$

3. $y = \frac{1+2x}{3x\sqrt{x}}$

Extension

[MAT 2013 1E]

The expression $\frac{d^2}{dx^2}[(2x-1)^4(1-x)^5] + \frac{d}{dx}[(2x+1)^4(3x^2-2)^2]$

is a polynomial of degree:

A) 9

B) 8

C) 7

D) less than 7

Ex 12E page267

Finding Equations of Tangents

Example

Find the equation of the **tangent** to the curve $y = x^2$ when $x = 3$.

Find the equation of the **normal** to the curve $y = x^2$ when $x = 3$.

Test your Understanding

Find the equation of the **normal** to the curve $y = x + 3\sqrt{x}$ when $x = 9$.

Extension

1. [STEP I 2005 Q2]

The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q. Show that R has coordinates $(pq, p + q)$.

The point S is the intersection of the normal to C at P and the normal to C at Q. If p and q are such that $(1,0)$ lies on the line PQ, show that S has coordinates $(p^2 + q^2 + 1, p + q)$, and that the quadrilateral $PSQR$ is a rectangle.

2. *STEP I 2012 Q4]*

The curve C has equation $xy = \frac{1}{2}$.

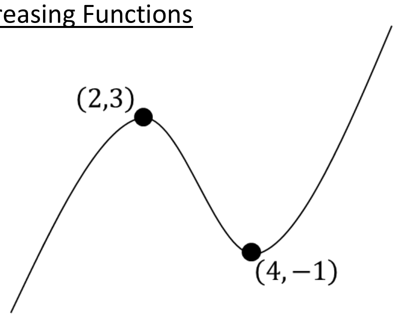
The tangents to C at the distinct points $P\left(p, \frac{1}{2p}\right)$ and $Q\left(q, \frac{1}{2q}\right)$, where p and q are positive, intersect at T and the normal to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right)$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

Ex 12F page269

Increasing and Decreasing Functions



Examples

1. Show that the function
$f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x.

2. Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x$ is decreasing.

Test Your Understanding

1. Show that the function
$f(x) = x^3 + 16x - 2$ is increasing for all real values of x.

2. Find the interval on which the function $f(x) = x^3 + 6x^2 - 135x$ is decreasing.

Ex 12G page271

Second Order Derivative

Examples

1. If $y = 3x^5 + \frac{4}{x^2}$, find $\frac{d^2y}{dx^2}$.

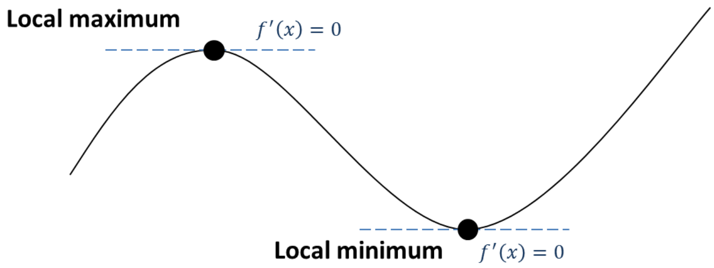
2. If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$.

Test you understanding

If $y = 5x^3 - \frac{x}{3\sqrt{x}}$, find $\frac{d^2y}{dx^2}$.

Ex 12H page272

Stationary Points/ Turning Points



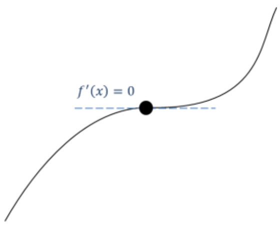
Examples

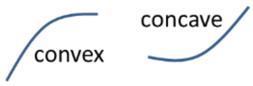
1. Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

2. Find the least value of $f(x) = x^2 - 4x + 9$

3. Find the turning point of $y = \sqrt{x} - x$

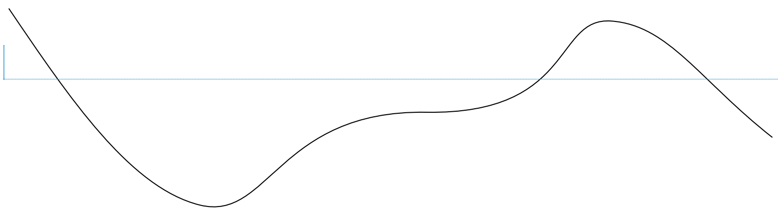
Points of Inflection





How do we tell what type of stationary point?

Method 1: Look at the gradient just before and just after the point



Commented [CL1]:

Commented [CL2R1]:

Local Minimum		
Gradient just before	Gradient at minimum	Gradient just after

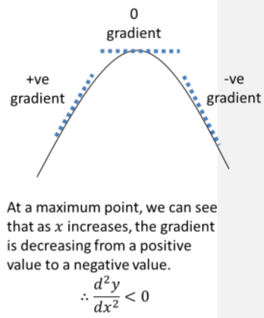
Point of Inflection		
Gradient before	Gradient at p.o.i	Gradient just after

Local Maximum		
Gradient just before	Gradient at maximum	Gradient just after

Example:

Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

Method 2: Using the second derivative



At a stationary point $x = a$:

- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

Example:
The stationary point of $y = x^4 - 32x$ is $(2, -48)$. Use the second derivative to classify this stationary point.

Test Your Understanding:

The curve with equation

$$y = x^2 - 32\sqrt{x} + 20, \qquad x > 0,$$

has a stationary point P.

Use calculus

(a) to find the coordinates of P,

(6)

(b) to determine the nature of the stationary point P.

(3)

<u>Sketching Graphs</u>

We can sketch graphs to help classify stationary points.

Example

By first finding the stationary points, sketch the graph of $y = \frac{1}{x} + 27x^3$

Extension

Extension

1 [MAT 2014 1C] The cubic
$y = kx^3 - (k+1)x^2 + (2-k)x - k$
has a turning point, that is a minimum, when $x = 1$ precisely for
A) $k > 0$
B) $0 < k < 1$
C) $k > \frac{1}{2}$
D) $k < 3$
E) all values of k

2 *[MAT 2004 1B]* The smallest value of the function:
$f(x) = 2x^3 - 9x^2 + 12x + 3$
In the range $0 \leq x \leq 2$ is what?

3 *[MAT 2001 1E]* The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occurs when:
A) $x = 0$
B) $x = 1 - \frac{1}{\sqrt{3}}$
C) $x = 1 + \frac{1}{\sqrt{3}}$
D) $x = 2\frac{1}{5}$

Hint: When two curves touch, their y values must match, but what else must also match?

4 [STEP I 2007 Q8] A curve is given by:
$y = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16)$
where a is a real number. Show that this curve touches the curve with equation $y = x^3$ at $(2,8)$. Determine the coordinates of any other point of intersection of the two curves.
(i) Sketch on the same axes these two curves when $a = 2$.
(ii) … when $a = 1$ (iii) when $a = -2$

Ex 12L page 276

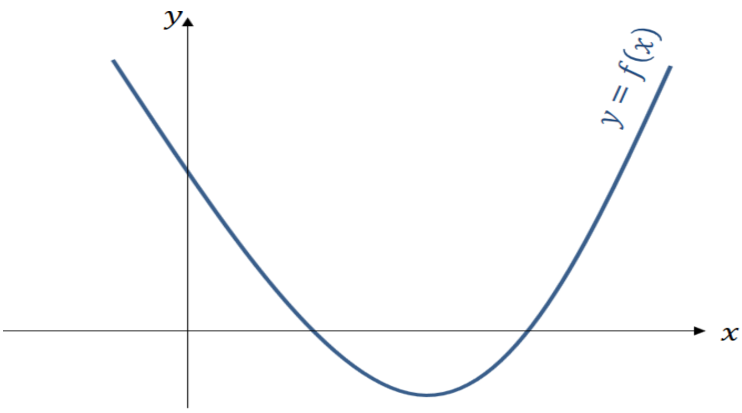
Sketching Gradient Functions

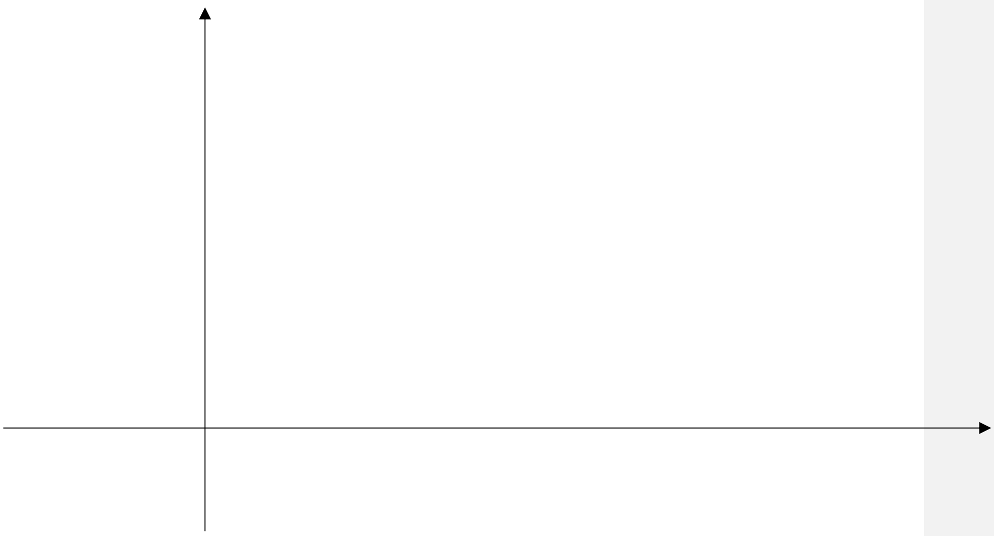
Example

Sketch the gradient function for the function $f(x) = x^2 + 3x + 2$

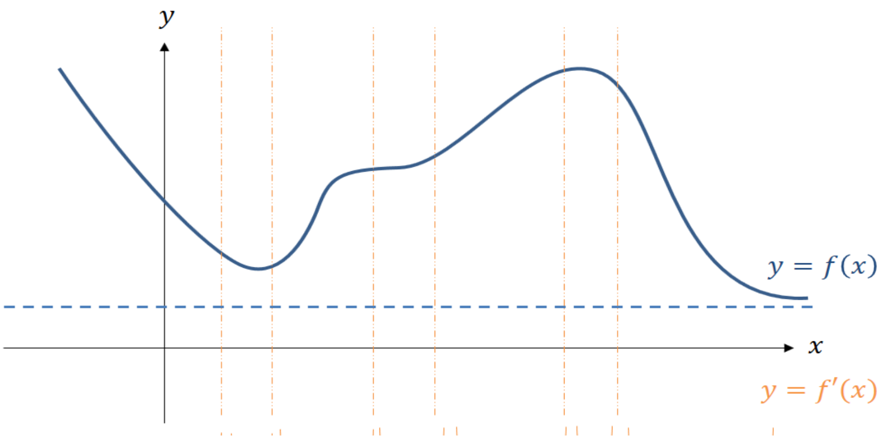
Sometimes **you won't be given the function explicitly**, you will only be given **the sketch**.

<u>Example</u>

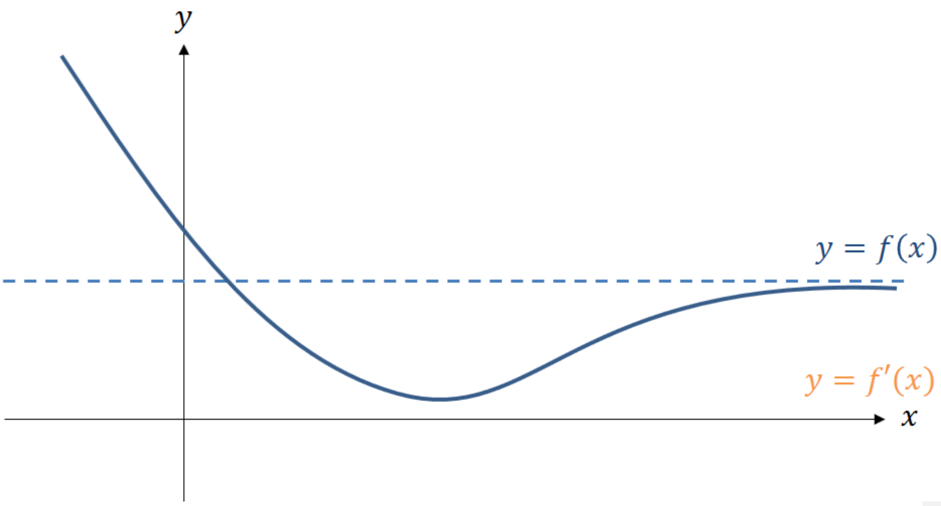


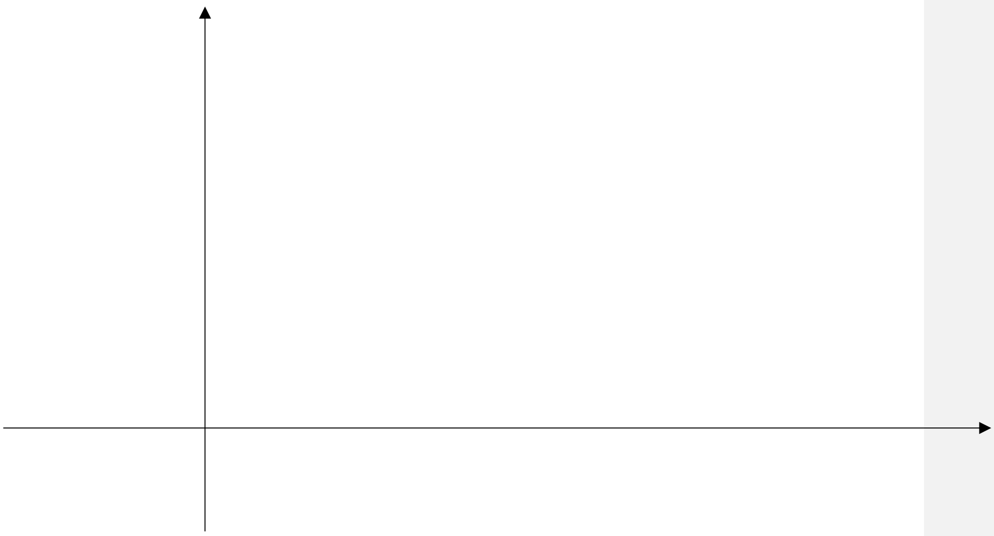


Example 2



Test Your Understanding





Summary

Y = f(x)	Y = f'(x)
max / min	Cuts the x - axis
Point of inflection	Touches the x – axis
Positive gradient	Above the x - axis
Negative gradient	Below the x - axis
Vertical asymptote	Vertical asymptote
Horizontal asymptote	Horizontal asymptote at x-axis

Extension

[MAT 2015 1B]

$$f(x) = (x + a)^n$$

where a is a real number and n is a positive whole number, and $n \geq 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will:

A) always be odd

B) always be even

C) depend on a but not n

D) depend on n but not a

E) depend on both a and n.

Ex 12J page 278

Optimisation Problems

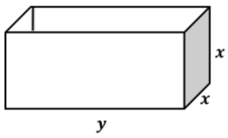
We can use differentiation to solve optimisation problems because it allows us to find maximum and minimum values of functions.

Optimisation problems in an exam usually follow the following pattern:

- There are 2 variables involved (you may have to introduce one yourself), typically lengths.
- There are expressions for **two different physical quantities**:
  - One is a **constraint**, e.g. "the surface area is 20cm^2".
  - The **other we wish to maximise/minimise**, e.g. "we wish to maximise the volume".
- We use the constraint to **eliminate one of the variables** in the latter equation, so that it is then **just in terms of one variable**, and we can then use differentiation to find the turning point.

Example

A large tank in the shape of a cuboid is to be made from $54m^2$ of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.



a) Show that the volume, V m^3, of the tank is given by

$$V = 18x - \frac{2}{3}x^3.$$

b) Given that x can vary, use differentiation to find the maximum or minimum value of V.

Test Your Understanding

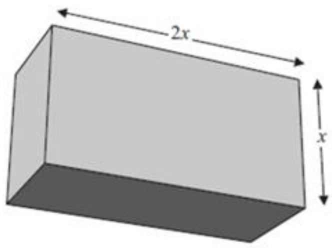


Figure 2

A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm, as shown in Figure 2.

The volume of the cuboid is 81 cubic centimetres.

(*a*) Show that the total length, L cm, of the twelve edges of the cuboid is given by

$$L = 12x + \frac{162}{x^2}.$$

(3)

(*b*) Use calculus to find the minimum value of L.

(6)

(*c*) Justify, by further differentiation, that the value of L that you have found is a minimum.

(2)

Extension

[STEP I 2006 Q2] A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze.

Prove that $A \leq 14\pi a^2$ and determine the minimum value of A.

Ex 12K page 281