Lower 6 Chapter 12

Differentiation

Chapter Overview

1. First Principles and finding the derivative of polynomials.
2. Find equations of tangents and normal to curves.
3. Identify increasing and decreasing functions.
4. Find and understand the second derivative $\frac{d^{2}y}{dx^{2}}$ or $f^{''}\left(x\right)$
5. Find stationary points and determine their nature.
6. Sketch a gradient function.
7. Model real-life problems.

  

The Gradient Function

Example

The point $A$ with coordinates $\left(4,16\right)$ lies on the curve with equation $y=x^{2}$.

At point $A$ the curve has gradient $g$.

1. Show that $g=\lim\_{h\to 0}\left(8+h\right)$
2. Deduce the value of $g$.

Example

Prove from first principles that the derivative of $x^{3}-2x=3x^{2}-2$

Test you understanding

Prove **from first principles** that the derivative of $x^{4}$ is $4x^{3}$.

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Differentiating $x^{n}$

Examples

1. $y=x^{5}$ 2. $f\left(x\right)=x^{\frac{1}{2}}$

3. $y=2x^{6}$ 4. $f\left(x\right)=\frac{x}{x^{4}}$

5. $y=\sqrt{x^{6}}$

Test Your Understanding

1. $y=x^{7}$ 2. $y=3x^{10}$

3. $f\left(x\right)=\frac{x^{\frac{1}{2}}}{x^{2}}$ 4. $y=ax^{a}$

5. $f(x)=\sqrt{49x^{7}}$

Differentiating Multiple Terms

Example

Differentiate $y=x^{2}+4x+3$

Questions

1. $y=2x^{2}-3x     $ 2. $y=4-9x^{3}$

3. $y=5x+1     $ 4. $y=ax              $(a is a constant)

5. $y=6x-3+px^{2}$ (p is a constant)

Harder Example

Let $f\left(x\right)=4x^{2}-8x+3$

1. Find the gradient of $y=f\left(x\right)$ at the point $\left(\frac{1}{2},0\right)$
2. Find the coordinates of the point on the graph of $y=f\left(x\right)$ where the gradient is 8.
3. Find the gradient of $y=f\left(x\right)$ at the points where the curve meets the line $y=4x-5$.

Test Your Understanding

Let $f\left(x\right)=x^{2}-4x+2$

1. Find the gradient of $y=f\left(x\right)$ at the point $\left(1,-1\right)$
2. Find the coordinates of the point on the graph of $y=f\left(x\right)$ where the gradient is 5.
3. Find the gradient of $y=f\left(x\right)$ at the points where the curve meets the line $y=2-x$.

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Differentiating Harder Expressions

1. Turn roots into powers $y=\sqrt{x}$

2. Split up fractions $y=\frac{x^{2}+3}{\sqrt{x}}$

3. Expand out brackets $y=x^{2}\left(x-3\right)$

4. Beware of numbers in denominators $y=\frac{1}{3x}$

Test your understanding

Differentiate the following

1. $y=\frac{1}{\sqrt{x}}$

2. $y=\frac{2+x^{3}}{x^{2}}$

3. $y=\frac{1+2x}{3x\sqrt{x}}$

Extension

 *[MAT 2013 1E]*

The expression $\frac{d^{2}}{dx^{2}}\left[\left(2x-1\right)^{4}\left(1-x\right)^{5}\right]+\frac{d}{dx}\left[\left(2x+1\right)^{4}\left(3x^{2}-2\right)^{2}\right]$

is a polynomial of degree:

1. 9
2. 8
3. 7
4. less than 7

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Finding Equations of Tangents

Example

Find the equation of the **tangent** to the curve $y=x^{2}$ when $x=3$.

Find the equation of the **normal** to the curve $y=x^{2}$ when $x=3$.

Test your Understanding

Find the equation of the **normal** to the curve $y=x+3\sqrt{x}$ when $x=9$.

Extension

*1. [STEP I 2005 Q2]*

The point $P$ has coordinates $\left(p^{2},2p\right)$ and the point $Q$ has coordinates $\left(q^{2},2q\right)$, where $p$ and $q$ are non-zero and $p\ne q$. The curve $C$ is given by $y^{2}=4x$. The point $R$ is the intersection of the tangent to $C$ at $P$ and the tangent to $C$ at $Q$. Show that $R$ has coordinates $\left(pq,p+q\right)$.

The point $S$ is the intersection of the normal to $C$ at $P$ and the normal to $C$ at $Q$. If $p$ and $q$ are such that $\left(1,0\right)$ lies on the line $PQ$, show that $S$ has coordinates $\left(p^{2}+q^{2}+1,p+q\right)$, and that the quadrilateral $PSQR$ is a rectangle.

2. *STEP I 2012 Q4]*

The curve $C$ has equation $xy=\frac{1}{2}$.

The tangents to $C$ at the distinct points $P\left(p,\frac{1}{2p}\right)$ and $Q\left(q,\frac{1}{2q}\right)$, where $p$ and $q$ are positive, intersect at $T$ and the normal to $C$ at these points intersect at $N$. Show that $T$ is the point

$$\left(\frac{2pq}{p+q},\frac{1}{p+q}\right)$$

In the case $pq=\frac{1}{2}$, find the coordinates of $N$. Show (in this case) that $T$ and $N$ lie on the line $y=x$ and are such that the product of their distances from the origin is constant.

Ex 12F page269

Increasing and Decreasing Functions

Examples

1. Show that the function
$f\left(x\right)=x^{3}+6x^{2}+21x+2$ is increasing for all real values of $x$.

2. Find the interval on which the function $f\left(x\right)=x^{3}+3x^{2}-9x$ is decreasing.

Test Your Understanding

1. Show that the function
$f\left(x\right)=x^{3}+16x-2$ is increasing for all real values of $x$.

2. Find the interval on which the function $f\left(x\right)=x^{3}+6x^{2}-135x$ is decreasing.

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Second Order Derivative

Examples

1. If $y=3x^{5}+\frac{4}{x^{2}}$, find $\frac{d^{2}y}{dx^{2}}$.

2. If $f\left(x\right)=3\sqrt{x}+\frac{1}{2\sqrt{x}}$, find $f''(x)$.

Test you understanding

If $y=5x^{3}-\frac{x}{3\sqrt{x}}$, find $\frac{d^{2}y}{dx^{2}}$.

Ex 12H page272

Stationary Points/ Turning Points



Examples

1. Find the coordinates of the turning points of $y=x^{3}+6x^{2}-135x$

2. Find the least value of $f\left(x\right)=x^{2}-4x+9$

3. Find the turning point of $y=\sqrt{x}-x$

Points of Inflection



How do we tell what type of stationary point?

Method 1: Look at the gradient just before and just after the point

|  |
| --- |
| **Local Maximum** |
| Gradient just before | Gradient at maximum | Gradient just after |
|   |  |  |

|  |
| --- |
| **Local Minimum** |
| Gradient just before | Gradient at minimum | Gradient just after |
|   |  |  |

|  |
| --- |
| **Point of Inflection** |
| Gradient before | Gradient at p.o.i | Gradient just after |
|   |  |  |

Example:

Find the stationary point on the curve with equation $y=x^{4}-32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

Method 2: Using the second derivative



At a stationary point $x=a$:

* If $f^{''}\left(a\right)>0$ the point is a local minimum.
* If $f^{''}\left(a\right)<0$ the point is a local maximum.
* If $f^{''}\left(a\right)=0$ it could be any type of point, so resort to Method 1.

Example:

The stationary point of $y=x^{4}-32x$ is $\left(2,-48\right)$. Use the second derivative to classify this stationary point.

Test Your Understanding:

Sketching Graphs

We can sketch graphs to help classify stationary points.

Example

By first finding the stationary points, sketch the graph of $y=\frac{1}{x}+27x^{3}$

Extension



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Sketching Gradient Functions

Example

Sketch the gradient function for the function $f\left(x\right)=x^{2}+3x+2$

Sometimes **you won’t be given the function explicitly**, you will only be given **the sketch**.

Example



Example 2





Test Your Understanding



Summary



Extension

[MAT 2015 1B]

$$f\left(x\right)=\left(x+a\right)^{n}$$

where $a$ is a real number and $n$ is a positive whole number, and $n\geq 2$. If $y=f(x)$ and $y=f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f^{'}\left(x\right)$ will:

1. always be odd
2. always be even
3. depend on $a$ but not $n$
4. depend on $n$ but not $a$
5. depend on both $a$ and $n$.

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Optimisation Problems

We can use differentiation to solve optimisation problems because it allows us to find maximum and minimum values of functions.

Optimisation problems in an exam usually follow the following pattern:

* There are 2 variables involved (you may have to introduce one yourself), typically lengths.
* There are expressions for **two different physical quantities**:
	+ One is a **constraint**, e.g. “the surface area is 20cm2”.
	+ The **other we wish to maximise/minimise**, e.g. “we wish to maximise the volume”.
* We use the constraint to **eliminate one of the variables** in the latter equation, so that it is then **just in terms of one variable**, and we can then use differentiation to find the turning point.

Example

A large tank in the shape of a cuboid is to be made from 54m2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is $x$ metres. Two of the opposite vertical faces are squares.

a) Show that the volume, V m3, of the tank is given by

 $V=18x-\frac{2}{3}x^{3}$.

b) Given that $x$ can vary, use differentiation to find the maximum or minimum value of $V$.

Test Your Understanding



Extension

[STEP I 2006 Q2] A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let $A$ be the area of the grass that the goat can graze.

Prove that $A\leq 14πa^{2}$ and determine the minimum value of $A$.

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