Lower 6 Chapter 10

Trigonometric Identities and Equations

Chapter Overview

1. Know exact trig values for $30°,\ 45°,\ 60°$ and understand unit circle.
2. Use identities $\frac{\sin x}{\cos x} \equiv \tan x$ and $\sin^2 x + \cos^2 x \equiv 1$
3. Solve equations of the form $\sin(n\theta) = k$ and $\sin(\theta \pm \alpha) = k$
4. Solve equations which are quadratic in sin/cos/tan.

Trigonometry	5.3	Understand and use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ Understand and use $\sin^2\theta + \cos^2\theta = 1$	These identities may be used to solve trigonometric equations or to prove further identities.
	5.4	Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.	Students should be able to solve equations such as $\sin(x + 70°) = 0.5$ for $0 < x < 360°$, $3 + 5\cos 2x = 1$ for $-180° < x < 180°$ $6\cos^2 x° + \sin x° - 5 = 0$, $0 \leqslant x < 360°$ giving their answers in degrees.

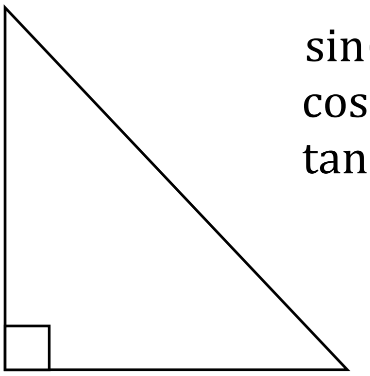
sin/cos/tan of $30°, 45°, 60°$

You will frequently encounter angles of $30°, 60°, 45°$ in geometric problems. Why?

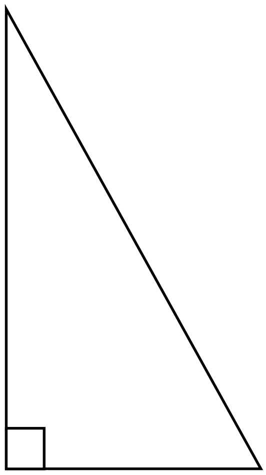
Although you will always have a calculator, you need to know how to derive these.

All you need to remember:

✏ **Draw half a unit square and half an equilateral triangle of side 2.**



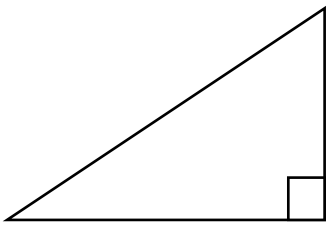
$\sin(45°) =$
$\cos(45°) =$
$\tan(45°) =$



$\sin(30°) =$
$\cos(30°) =$
$\tan(30°) =$
$\sin(60°) =$
$\cos(60°) =$
$\tan(60°) =$

The Unit Circle and Trigonometry

For values of θ in the range $0 < \theta < 90°$, you know that $\sin\theta$ and $\cos\theta$ are lengths on a right-angled triangle:



And what would be the **gradient** of the bold line (hypotenuse)?

But how do we get the rest of the graph for sin, cos and tan when $90° \leq \theta \leq 360°$?

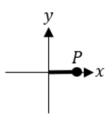
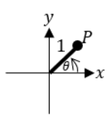
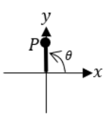
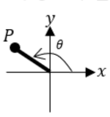
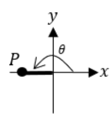
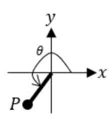
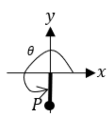
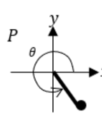
The point P on a unit circle, such that OP makes an angle θ with the positive x-axis, has coordinates $(\cos\theta, \sin\theta)$.
OP has gradient $\tan\theta$.

Angles are always measured **anticlockwise**.

We can consider the coordinate P $(\cos\theta, \sin\theta)$ as θ increases from 0 to $360°$…

Use the unit circle to determine each value in the table, **using either "0", "+ve", "-ve", "1", "-1" or "undefined"**.

x-value → $\cos\theta$; y-value → $\sin\theta$; Gradient of OP. → $\tan\theta$

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta = 0$ 			
$0 < \theta < 90°$ 			
$\theta = 90°$ 			
$90° < \theta < 180°$ 			
$\theta = 180°$ 			
$180° < \theta < 270°$ 			
$\theta = 270°$ 			
$270° < \theta < 360°$ 			

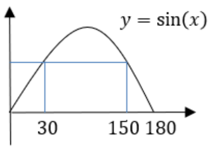
The unit circle explains the behaviour of the trigonometric graphs beyond 90°.
However, the easiest way to remember whether $\sin(x), \cos(x), \tan(x)$ are positive or negative is to just do a **very quick sketch (preferably mentally!)** of the corresponding graph.
Note: The textbook uses something called '*CAST diagrams*'. We will not be using them in this booklet, but you may wish to look at this technique as an alternative approach to various problems in the chapter.

A Few Trigonometric Angle Laws

The following are all easily derivable using a quick sketch of a trigonometric graph, and are merely a convenience so you don't always have to draw out a graph every time.
You are highly encouraged to **memorise these** so that you can do exam questions faster.

1. $\sin(x) = \sin(180° - x)$

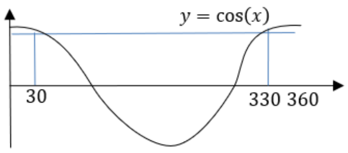
   e.g. $\sin(150°) = \sin(30°)$



We saw this in the previous chapter when covering the 'ambiguous case' when using the sine rule.

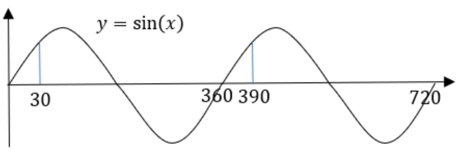
2. $\cos(x) = \cos(360° - x)$

   e.g. $\cos(330°) = \cos(30°)$



3. *sin* and *cos* repeat every 360° but *tan* every 180°

   e.g. $\sin(390°) = \sin(30°)$



4. $\sin(x) = \cos(90° - x)$

   e.g. $\sin(50°) = \cos(40°)$

Remember from the previous chapter that "cosine" by definition is the sine of the "complementary" angle. This was/is never covered in the textbook but caught everyone by surprise when it came up in a C3 exam.

Examples

Without a calculator, work out the value of each below:

$\tan(225°) =$

$\tan(210°) =$

$\sin(150°) =$

$\cos(300°) =$

$\sin(-45°) =$

$\cos(750°) =$

$\cos(120°) =$

$\cos(315°) =$

$\sin(420°) =$

$\tan(-120°) =$

$\tan(-45°) =$

$\sin(135°) =$

Exercise 10A and B Pg 207 and 209

Trigonometric identities

$$\boldsymbol{tan\,\theta = \frac{sin\,\theta}{cos\,\theta}}$$

$$\boldsymbol{sin^2\theta + cos^2\theta = 1}$$

Examples

Prove that $1 - \tan\theta \sin\theta \cos\theta \equiv \cos^2\theta$

Prove that $\tan\theta + \frac{1}{\tan\theta} \equiv \frac{1}{\sin\theta\cos\theta}$

Simplify $5 - 5\sin^2\theta$

Test your understanding

Prove that $\frac{\tan x \cos x}{\sqrt{1-\cos^2 x}} \equiv 1$

Prove that $\frac{\cos^4 \theta - \sin^4 \theta}{\cos^2 \theta} \equiv 1 - \tan^2 \theta$

Prove that $\tan^2\theta \equiv \frac{1}{\cos^2\theta} - 1$

Exercise 10C Pg 211/212

Extension:

[MAT 2008 1C] The simultaneous equations in x, y,

$$(\cos\theta)x - (\sin\theta)y = 2$$
$$(\cos\theta)x + (\sin\theta)y = 1$$

are solvable:

A) for all values of θ in range $0 \le \theta < 2\pi$
B) except for one value of θ in range $0 \le \theta < 2\pi$
C) except for two values of θ in range $0 \le \theta < 2\pi$
D) except for three values of θ in range $0 \le \theta < 2\pi$

Solving trigonometric equations

Solve $\sin\theta = \frac{1}{2}$ in the interval $0 \leq \theta \leq 360°$.

Solve $5\tan\theta = 10$ in the interval $-180° \leq \theta < 180°$

Solve $\sin\theta = -\frac{1}{2}$ in the interval $0 \leq \theta \leq 360°$.

Solve $\sin\theta = \sqrt{3}\cos\theta$ in the interval $0 \leq \theta \leq 360°$.

Solve $2\cos\theta = \sqrt{3}$ in the interval $0 \leq \theta \leq 360°$.

Solve $\sqrt{3}\sin\theta = \cos\theta$ in the interval $-180° \leq \theta \leq 180°$.

Harder equations

Solve $\cos 3x = -\frac{1}{2}$ in the interval $0 \leq x \leq 360°$.

Solve $\sin(2x + 30°) = \frac{1}{\sqrt{2}}$ in the interval $0 \leq x \leq 360°$.

Solve $\sin x = 2\cos x$ in the interval $0 \leq x < 300°$

Solve, for $0 \leq x < 180°$,

$$\cos(3x - 10°) = -0.4,$$

giving your answers to 1 decimal place. You should show each step in your working.

(7)

Exercise 10E Pg 218/219

Quadratics in sin/cos/tan

Solve $5\sin^2 x + 3\sin x - 2 = 0$ in the interval $0 \leq x \leq 360°$.

Solve $\tan^2 \theta = 4$ in the interval $0 \leq x \leq 360°$.

Solve $2\cos^2 x + 9\sin x = 3\sin^2 x$ in the interval $-180° \leq x \leq 180°$.

(*a*) Show that the equation

$$5\sin x = 1 + 2\cos^2 x$$

can be written in the form

$$2\sin^2 x + 5\sin x - 3 = 0.$$

(2)

(*b*) Solve, for $0 \leq x < 360°$,

$$2\sin^2 x + 5\sin x - 3 = 0.$$

(4)

Exercise 10F Pg 221/222

Extension

1 *[MAT 2010 1C]* In the range $0 \leq x < 360°$, the equation

$$\sin^2 x + 3 \sin x \cos x + 2 \cos^2 x = 0$$

Has how many solutions?

2 *[MAT 2014 1E]* As x varies over the real numbers, the largest value taken by the function $(4 \sin^2 x + 4 \cos x + 1)^2$ equals what?