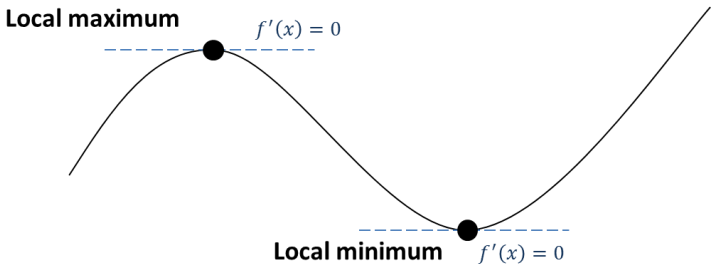
Stationary Points/ Turning Points



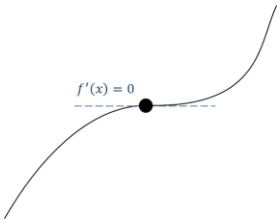
Examples

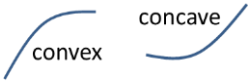
1. Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

2. Find the least value of $f(x) = x^2 - 4x + 9$

3. Find the turning point of $y = \sqrt{x} - x$

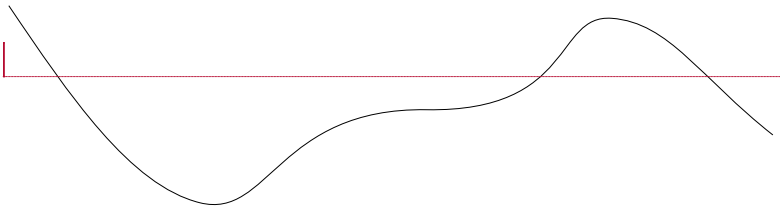
Points of Inflection





How do we tell what type of stationary point?

Method 1: Look at the gradient just before and just after the point



Commented [CL1]:

Commented [CL2R1]:

Local Minimum		
Gradient just before	Gradient at minimum	Gradient just after

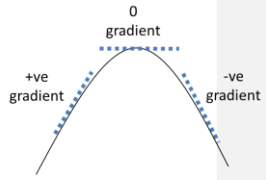
Point of Inflection		
Gradient before	Gradient at p.o.i	Gradient just after

Local Maximum		
Gradient just before	Gradient at maximum	Gradient just after

Example:

Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

Method 2: Using the second derivative



At a maximum point, we can see that as x increases, the gradient is decreasing from a positive value to a negative value.

$$\therefore \frac{d^2y}{dx^2} < 0$$

At a stationary point $x = a$:

- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

Example:
The stationary point of $y = x^4 - 32x$ is $(2, -48)$. Use the second derivative to classify this stationary point.

Test Your Understanding:

The curve with equation

$$y = x^2 - 32\sqrt{x} + 20, \quad x > 0,$$

has a stationary point P.

Use calculus

(a) to find the coordinates of P,

(6)

(b) to determine the nature of the stationary point P.

(3)

Sketching Graphs

We can sketch graphs to help classify stationary points.

Example

By first finding the stationary points, sketch the graph of $y = \frac{1}{x} + 27x^3$

Extension

Extension

1 [MAT 2014 1C] The cubic $y = kx^3 - (k+1)x^2 + (2-k)x - k$ has a turning point, that is a minimum, when $x = 1$ precisely for

A) $k > 0$
B) $0 < k < 1$
C) $k > \frac{1}{2}$
D) $k < 3$
E) all values of k

2 *[MAT 2004 1B]* The smallest value of the function:

$f(x) = 2x^3 - 9x^2 + 12x + 3$

In the range $0 \le x \le 2$ is what?

3 *[MAT 2001 1E]* The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \le x \le 2\frac{1}{5}$ occurs when:

A) $x = 0$
B) $x = 1 - \frac{1}{\sqrt{3}}$
C) $x = 1 + \frac{1}{\sqrt{3}}$
D) $x = 2\frac{1}{5}$

Hint: When two curves touch, their y values must match, but what else must also match?

4 [STEP I 2007 Q8] A curve is given by:

$y = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16)$

where a is a real number. Show that this curve touches the curve with equation $y = x^3$ at $(2,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes these two curves when $a = 2$.

(ii) ... when $a = 1$ (iii) when $a = -2$