# Upper 6 Chapter 2

## Functions and Graphs

### Chapter Overview

1. The Modulus Function

2. Mappings vs Functions, Domain and Range

3. Composite Functions

4. Inverse Functions

5. Transformations of the form $y = |f(x)|$ or $y = f(|x|)$. Combined transformations and transforming the modulus function.

6. Solving modulus problems

| | | | |
|---|---|---|---|
| **2**<br>**Algebra and functions**<br>*continued* | 2.7 | **Understand and use graphs of functions; sketch curves defined by simple equations including polynomials** | **Graph to include simple cubic and quartic functions,**<br>**e.g. sketch the graph with equation $y = x^2(2x - 1)^2$** |
| | | The modulus of a linear function. | Students should be able to sketch the graphs of $y = \lvert ax + b \rvert$<br>They should be able to use their graph.<br>For example, sketch the graph with equation $y = \lvert 2x - 1 \rvert$ and use the graph to solve the equation $\lvert 2x - 1 \rvert = x$ or the inequality $\lvert 2x - 1 \rvert > x$ |
| | | $y = \frac{a}{x}$ **and** $y = \frac{a}{x^2}$<br>**(including their vertical and horizontal asymptotes)** | **The asymptotes will be parallel to the axes e.g. the asymptotes of the curve with equation $y = \frac{2}{x + a} + b$ are the lines with equations $y = b$ and $x = -a$** |
| | | **Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.** | |
| | | **Understand and use proportional relationships and their graphs.** | **Direct proportion between two variables.**<br>**Express relationship between two variables using proportion "$\propto$" symbol or using equation involving constant**<br>**e.g. the circumference of a semicircle is directly proportional to its diameter so $C \propto d$ or $C = kd$ and the graph of $C$ against $d$ is a straight line through the origin with gradient $k$.** |

| | | | |
|---|---|---|---|
| **2**<br>**Algebra and functions**<br>*continued* | 2.8 | Understand and use composite functions; inverse functions and their graphs. | The concept of a function as a one-one or many-one mapping from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$. The notation $f : x \mapsto$ and $f(x)$ will be used. Domain and range of functions.<br>Students should know that fg will mean 'do g first, then f' and that if $f^{-1}$ exists, then<br>$f^{-1}f(x) = ff^{-1}(x) = x$<br>They should also know that the graph of<br>$y = f^{-1}(x)$ is the image of the graph of<br>$y = f(x)$ after reflection in the line $y = x$ |
| | 2.9 | **Understand the effect of simple transformations on the graph of $y = f(x)$, including sketching associated graphs:**<br>$y = af(x)$, $y = f(x) + a$,<br>$y = f(x + a)$, $y = f(ax)$<br>and combinations of these transformations | Students should be able to find the graphs of $y = \lvert f(x) \rvert$ and $y = \lvert f(-x) \rvert$, given the graph of $y = f(x)$.<br>Students should be able to apply a combination of these transformations to any of the functions in the A Level specification (quadratics, cubics, quartics, reciprocal, $\frac{a}{x^2}$, $\lvert x \rvert$, $\sin x$, $\cos x$, $\tan x$, $e^x$ and $a^x$) and sketch the resulting graph.<br>Given the graph of $y = f(x)$, students should be able to sketch the graph of, e.g. $y = 2f(3x)$, or $y = f(-x) + 1$,<br>and should be able to sketch (for example)<br>$y = 3 + \sin 2x$, $y = -\cos\left(x + \frac{\pi}{4}\right)$ |

| | | | |
|---|---|---|---|
| **2**<br>**Algebra and functions**<br>*continued* | 2.11 | Use of functions in modelling, including consideration of limitations and refinements of the models. | For example, use of trigonometric functions for modelling tides, hours of sunlight, etc. Use of exponential functions for growth and decay (see Paper 1, Section 6.7). Use of reciprocal function for inverse proportion (e.g. pressure and volume). |

# The Modulus Function

Example:

1. If $f(x) = |2x - 3| + 1$, find

a) $f(5)$

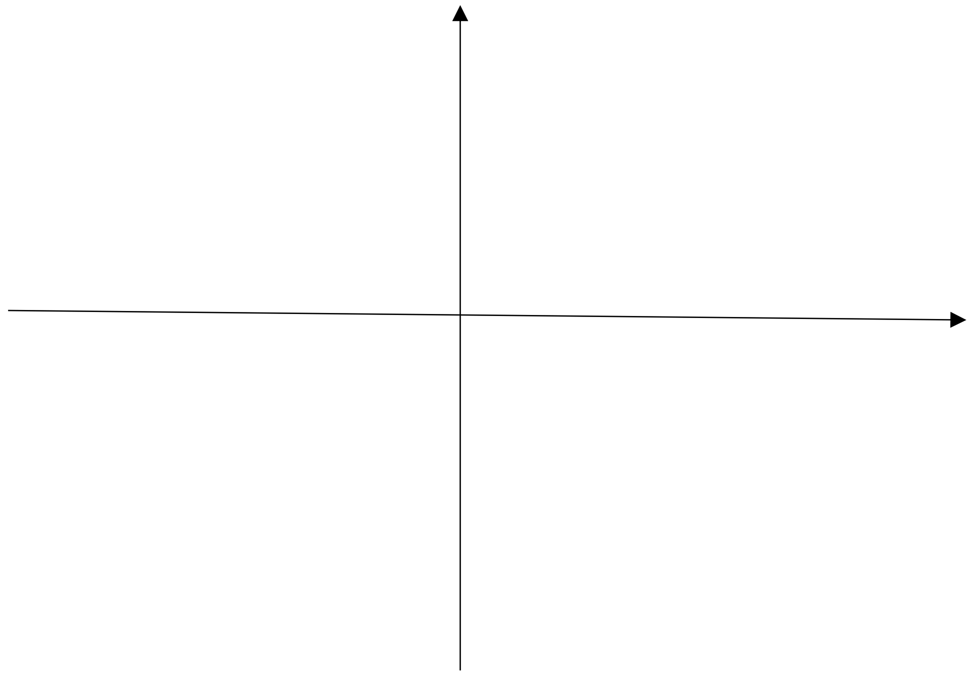
b) $f(-2)$

c) $f(1)$

<u>Modulus Graphs</u>

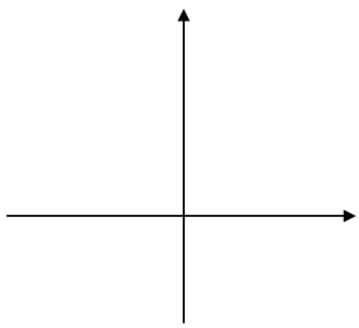
$y = |x|$

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

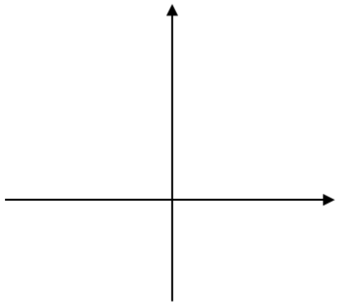


Examples

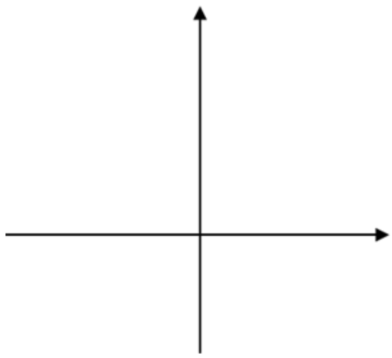
1. Sketch $y = |2x - 3|$



2. Solve $|2x - 3| = 5$

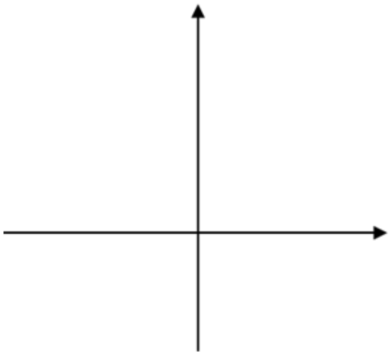


3. Solve $|3x - 5| = 2 - \frac{1}{2}x$

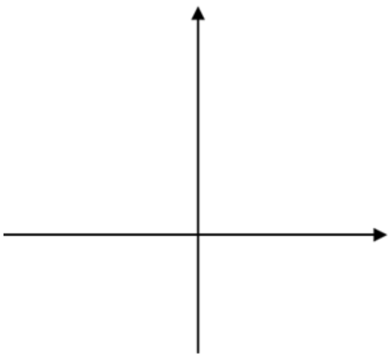


Test Your Understanding

1. Solve $|x + 1| = 2x + 5$



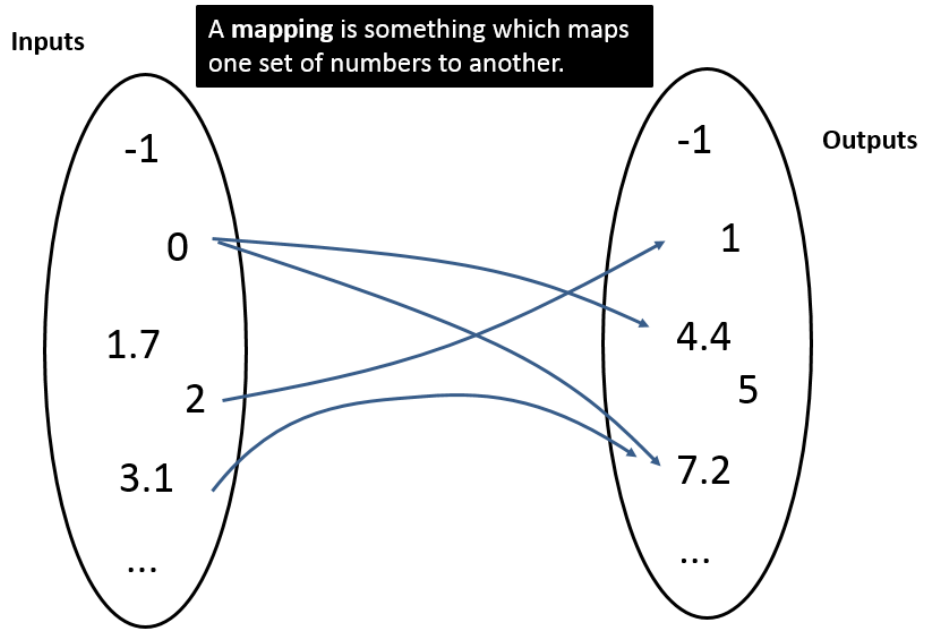
2. Solve $|4x - 1| < 2x$



Ex 2a page 18

## Mappings

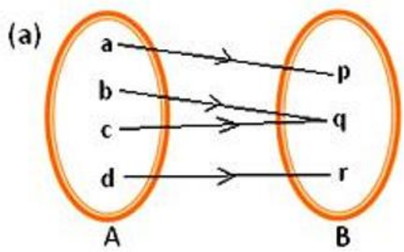
A **mapping** is something which maps one set of numbers to another.



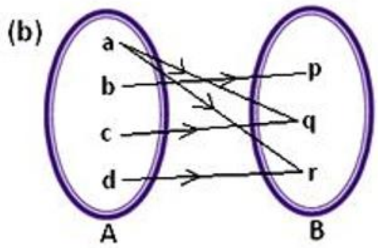

✏ The **domain** is the set of possible inputs.

✏ The **range** is the set of possible outputs.

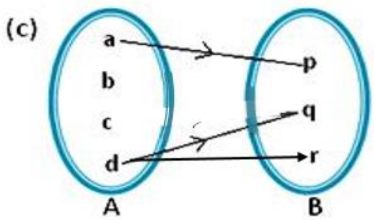
## Types of Mappings



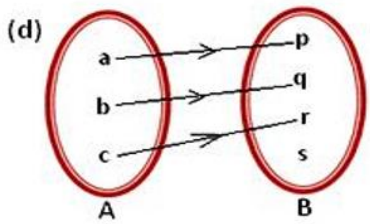

Many to one




Many to many




One to many



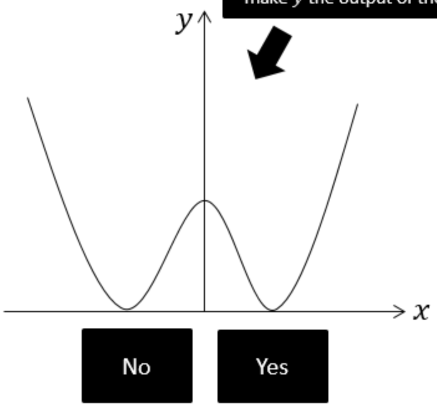

One to one

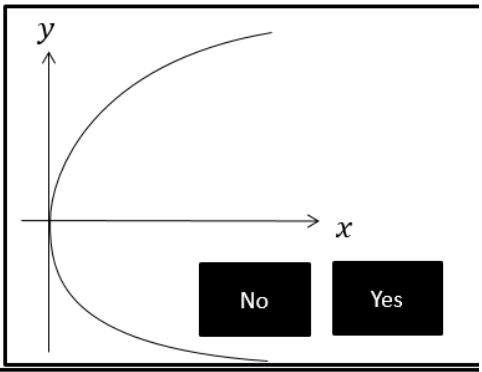
Functions

| Type | Description | Example |
|---|---|---|
| Many-to-one Function | | |
| One-to-one Function | | |

<u>Which of these are functions?</u>

**Note:** We can illustrate a mapping/function graphically, by plotting a point $(x, y)$ if $x$ maps to $y$. For this reason we write $y = f(x)$ to mean "make $y$ the output of the function".




$f(x) = \sqrt{x}$ Domain: $x \in \mathbb{R}$

No Yes

$f(x) = 2^x$ Domain: $x \in \mathbb{R}$
(i.e. all real values)

No Yes

$f(x) = \pm\sqrt{x}$ Domain: $x \geq 0$

No Yes

<u>Domain and Range</u>

Remember:

The **domain** is ………………………………………………………………………………………………

The **range** is ………………………………………………………………………………………………..

Using a sketch, try to identify the range of the following functions.

1. $f(x) = x^2, \quad x \in \mathbb{R}$

2. $f(x) = \frac{1}{x}, \quad x \in \mathbb{R}, x \neq 0$

3. $f(x) = \ln x\,, \quad x \in \mathbb{R}, x > 0$

4. $f(x) = e^x, \quad x \in \mathbb{R},$

5. $f(x) = x^2 + 2x + 9, \quad x \in \mathbb{R}$

6. $f(x) = x^2, \quad x \in \mathbb{R}, \ -1 \leq x \leq 4$

Further Example

Find the range of each of the following functions.

a) $f(x) = 3x - 2$, domain $\{1,2,3,4\}$

b) $g(x) = x^2$, domain $\{x \in \mathbb{R},\ -5 \leq x \leq 5\}$

c) $h(x) = \frac{1}{x}$, domain $\{x \in \mathbb{R},\ 0 < x \leq 3\}$

State if the functions are one-to-one or many-to-one.

# Piecewise Functions

Examples

1. The function $f(x)$ is defined by

$$f: x \to \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.

b) Solve $f(x) = 19$

2.

The function s is defined by

$$s(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x, & x \geqslant 0 \end{cases}$$

**a** Sketch $y = s(x)$.

**b** Find the value(s) of $a$ such that $s(a) = 43$.

**c** Solve $s(x) = x$.

Test Your Understanding

1. The function $f$ is defined by

$$f: x \to e^x + 2, \quad x \in \mathbb{R}$$

State the range of $f$.

2. The function $g$ is defined by

$$g: x \to x^2 - 4x + 1, \quad x \in \mathbb{R},\ 0 \leq x \leq 5$$

Find the range of $g$.

Ex 2B Pg 30

## Composite Functions

## Examples

1. Let $f(x) = x^2 + 1$, and $g(x) = 4x - 2$. Find

$a)\ fg(2)$

$b)\ fg(x)$

$c)\ gf(x)$

$d)\ f^2(x)$

Solve $gf(x) = 38$

2. The functions $f$ and $g$ are defined by

$$f: x \to |2x - 8| g: x \to \frac{x+1}{2}$$

a) Find $fg(3)$

b) Solve $fg(x) = x$

Test your understanding

1. The functions and $f$ and $g$ are defined by

$$f: x \to 2|x| + 3, \quad x \in \mathbb{R}$$
$$g: x \to 3 - 4x, \quad x \in \mathbb{R}$$

a) Find $fg(1)$

b) Solve the equation

$$gg(x) + [g(x)]^2 = 0$$

2. The functions $f$ and $g$ are defined by

$$f: x \to e^x + 2, \quad x \in \mathbb{R} g: x \to \ln x, \qquad x > 0$$

a) Find $fg(x)$, giving your answer in its simplest form.

Extension

[MAT 2014 1F]
The functions $S$ and $T$ are defined for real numbers by $S(x) = x + 1$ and $T(x) = -x$.
The function $S$ is applied $s$ times and the function $T$ is applied $t$ times, in some order, to produce the function

$$F(x) = 8 - x$$

It is possible to deduce that:

i) $s = 8$ and $t = 1$
ii) $s$ is odd and $t$ is even.
iii) $s$ is even and $t$ is odd.
iv) $s$ and $t$ are powers of 2.
v) none of the above.

[MAT 2012 Q2]
Let $f(x) = x + 1$ and $g(x) = 2x$.
i) Show that $f^2g(x) = gf(x)$
ii) Note that $gf^2g(x) = 4x + 4$
Find all the other ways of combining $f$ and $g$ that result in the function $4x + 4$.
iii) Let $i, j, k \geq 0$ be integers. Determine the function

$$f^i g f^j g f^k(x)$$

iv) Let $m \geq 0$ be an integer. How many different ways of combining the functions $f$ and $g$ are there that result in the function $4x + 4m$?

Ex 2C Pg 34

Inverse Functions

| |
|---|
| |

Why must the function be one-to-one for an inverse function to exist?

How do we find an inverse function?

Example

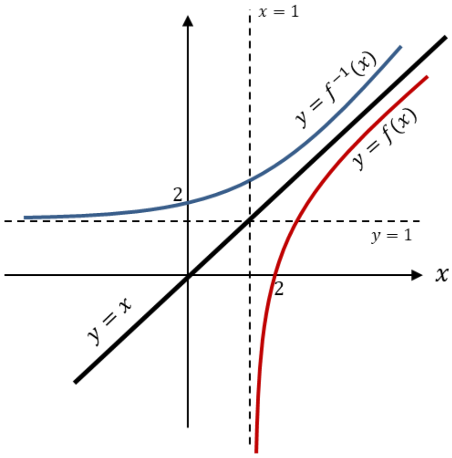
Steps

1. If $f(x) = 3 - 4x$, find $f^{-1}(x)$

2. If $f(x) = \frac{x+2}{2x-1}$, $x \neq \frac{1}{2}$, determine $f^{-1}(x)$

## Graphing an Inverse Function

The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.




Example

If $g(x)$ is defined as $g(x) = \sqrt{x-2}\ \{x \in \mathbb{R}, x \geq 2\}$

a) Find the range of $g(x)$.

b) Calculate $g^{-1}(x)$

c) Sketch the graphs of both functions.

d) State the domain and range of $g^{-1}(x)$.

## Test your understanding

The function $f$ is defined by

$$f: x \to e^x + 2, \quad x \in \mathbb{R}$$

(a) Find $f^{-1}$, the inverse function of $f$, stating its domain.

(b) On the same axe sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes.

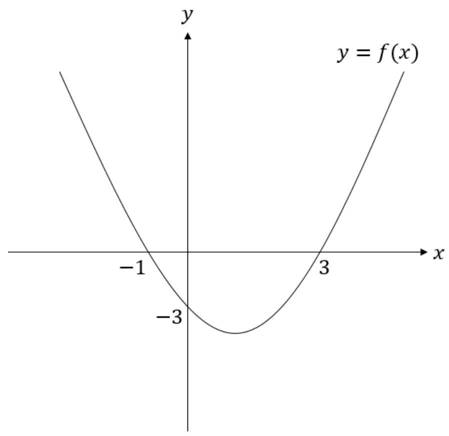
Ex 2D Pg 38

<u>Sketching $y = |f(x)|$ and $y = f(|x|)$</u>

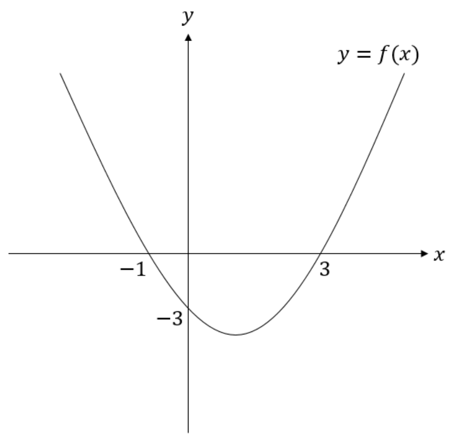
It is important to understand the difference between $y = |f(x)|$ and $y = f(|x|)$ and to be able to graph each of these.

The graph below shows $y = f(x)$ where $f(x) = (x - 3)(x + 1)$. Sketch $y = |f(x)|$ and $y = f(|x|)$

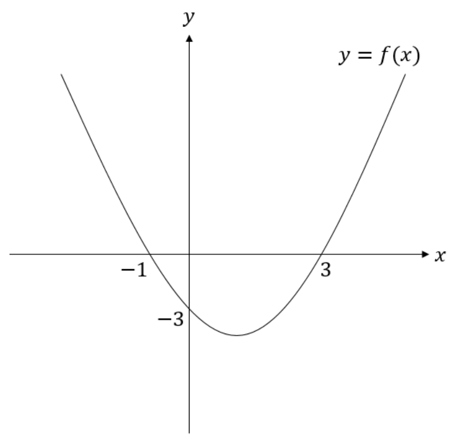
$y = f(x)$




$y = |f(x)|$




$y = f(|x|)$



Test your understanding

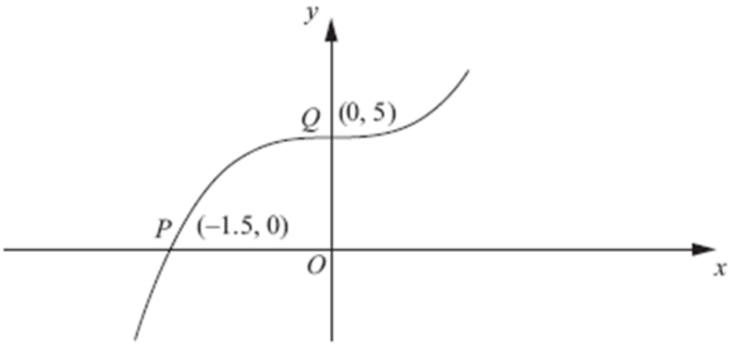



Figure 2 shows part of the curve with equation $y = \mathrm{f}(x)$.
The curve passes through the points $P(-1.5, 0)$ and $Q(0, 5)$ as shown.

On separate diagrams, sketch the curve with equation

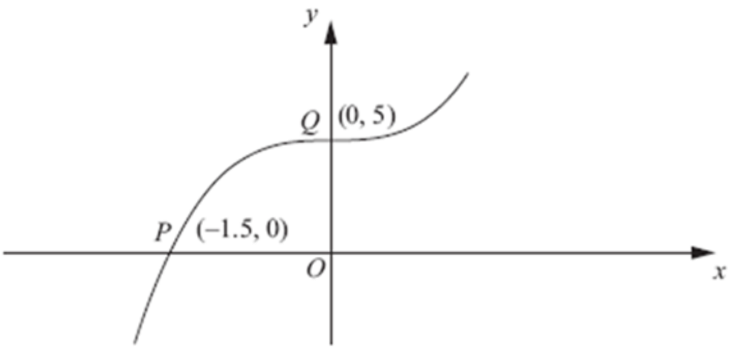
(*a*) $y = |\mathrm{f}(x)|$

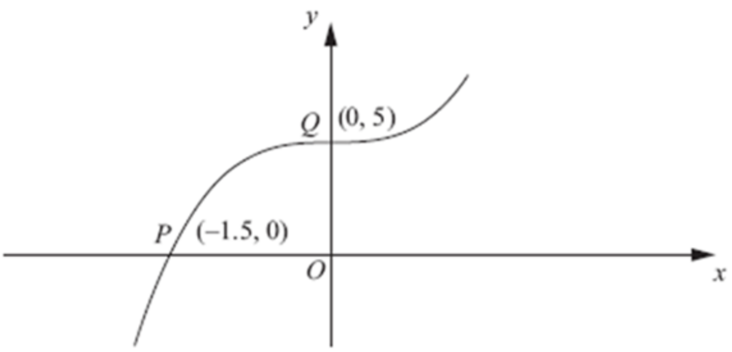
**(2)**

(*b*) $y = \mathrm{f}(|x|)$

**(2)**

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.



## Test your understanding

Sketch for $-2\pi \leq x \leq 2\pi$:

a) $y = |\sin(x)|$
b) $y = \sin(|x|)$

## Extension

[SMC 2008 Q25] What is the area of the polygon forms by all the points $(x, y)$ in the plane satisfying the inequality $\big||x| - 2\big| + \big||y| - 2\big| \leq 4$ ?
A 24 B 32 C 64 D 96 E 112

Ex 2E Pg 42

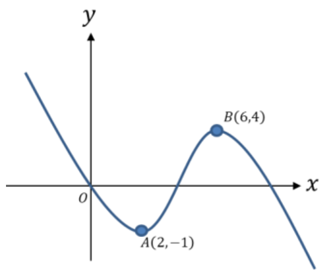
<u>Combining Transformations</u>

| | Affects what axis? | What we expect or opposite? |
|---|---|---|
| Change inside f( ) | | |
| Change outside f( ) | | |

In L6 we studied transformations. Here we are asked to combine more than one transformation to a function.

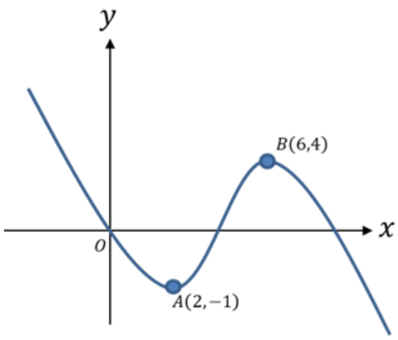
Examples

1. Here is a graph of $y = f(x)$.

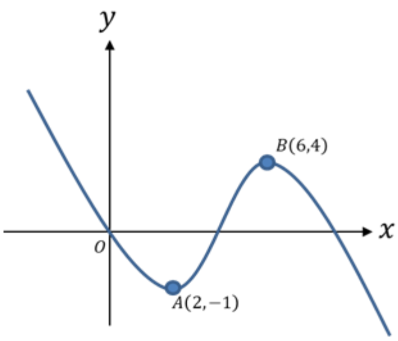



Sketch the graph of:

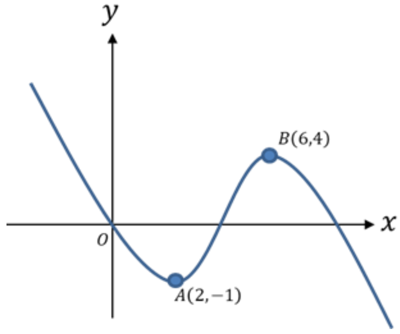
a) $y = 2f(x+2)$




b) ) $y = -f(2x)$



c) $y = |f(-x)|$




Test Your Understanding

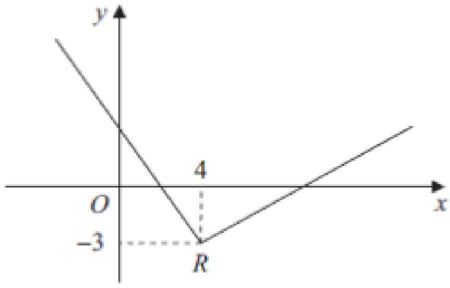


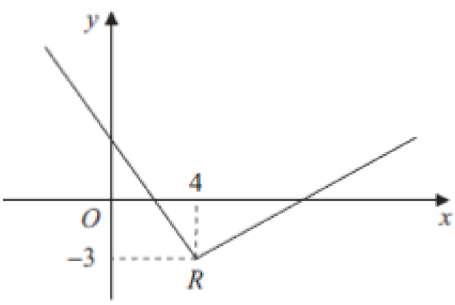

**Figure 1**

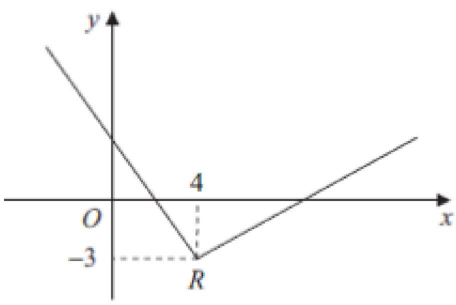
Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point $R\ (4, -3)$, as shown in Figure 1.

Sketch, on separate diagrams, the graphs of

(*a*) $y = 2\mathrm{f}(x + 4)$, **(3)**
(*b*) $y = |\mathrm{f}(-x)|$ . **(3)**
On each diagram, show the coordinates of the point corresponding to *R*.




Ex2F Pg 47

<u>Solving Modulus Problems</u>

Modulus questions are very common in exams. It is important that you are confident with all aspects of questions including sketching graphs, finding the range ad domain and solving equations.

Examples

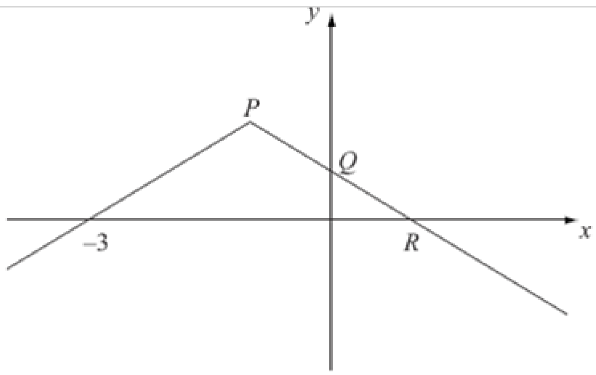
1. Given the function $f(x) = 3|x - 1| - 2,\ x \in \mathbb{R}$,

   (a) Sketch the graph of $y = f(x)$

   (b) State the range of $f$.

   (c) Solve the equation $f(x) = \frac{1}{2}x + 3$

Test Your Understanding




Given that $f(x) = 2 - |x + 1|$,

(c) find the coordinates of the points $P$, $Q$ and $R$, **(3)**

(d) solve $f(x) = \frac{1}{2}x$. **(5)**

Extension

*[MAT 2006 1I]*

The equation $|x| + |x - 1| = 0$ has how many solutions?

Ex 2G Pg 51